Name ______________________ Class ______________ Date ________

13.3 The Base *e*

Essential Question: How is the graph of $g(x) = ae^{x-h} + k$ related to the graph of $f(x) = e^x$?

Resource Locker

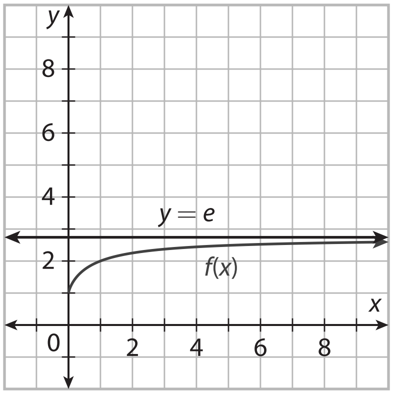
Explore 1 Graphing and Analyzing $f(x) = e^x$

The following table represents the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ for several values of x.

x	1	10	100	1000	...
$f(x)$	2	2.5937 ...	2.7048 ...	2.7169 ...	...

As the value of x increases without bound, the value of $f(x)$ approaches a number whose decimal value is 2.718… This number is irrational and is called e. You can write this in symbols as $f(x) \to e$ as $x \to +\infty$.

If you graph $f(x)$ and the horizontal line $y = e$, you can see that $y = e$ is the horizontal asymptote of $f(x)$.

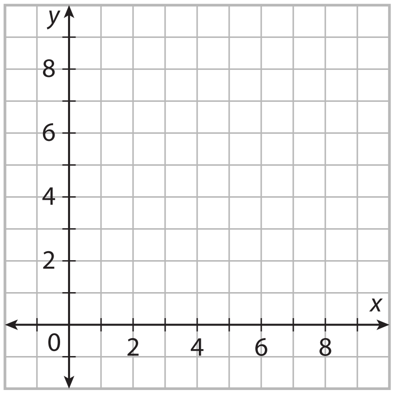


Even though e is an irrational number, it can be used as the base of an exponential function. The number e is sometimes called the natural base of an exponential function and is used extensively in scientific and other applications involving exponential growth and decay.

Ⓐ Fill out the table of values below for the function $f(x) = e^x$. Use decimal approximations.

x	−10	−1	−0.5	0	0.5	1	1.5	2
$f(x) = e^x$	4.54×10^{-5}	$\frac{1}{e} = 0.367...$	0.606...		$\sqrt{e} =$			

Ⓑ Plot the points on a graph.



Ⓒ The domain of $f(x) = e^x$ is $\{x \mid \quad\}$.

The range of $f(x) = e^x$ is $\{y \mid \quad\}$.

Ⓓ Is the function increasing or decreasing? For what values of x is it increasing/decreasing?

Ⓔ The function's y-intercept is $\left(0, \boxed{}\right)$ because $f(0) = e^0 = \boxed{}$ and $x = 0$ is in the domain of the function.

Ⓕ Another point on the graph that can be used as a reference point is $\left(1, \boxed{}\right)$.

Ⓖ Identify the end behavior.

$f(x) \rightarrow \boxed{}$ as $x \rightarrow \infty$

$f(x) \rightarrow \boxed{}$ as $x \rightarrow -\infty$

There is a horizontal asymptote at $y = \boxed{}$.

Reflect

1. What is the relationship between the graphs of $f(x) = e^x$, $g(x) = 2^x$, and $h(x) = 3^x$? (Hint: Sketch the graphs on your own paper.)

Explain 1 Graphing Combined Transformations of $f(x) = e^x$

When graphing combined transformations of $f(x) = e^x$ that result in the function $g(x) = a \cdot e^{x-h} + k$, it helps to focus on two reference points on the graph of $f(x)$, $(0, 1)$ and $(1, e)$, as well as on the asymptote $y = 0$. The table shows these reference points and the asymptote $y = 0$ for $f(x) = e^x$ and the corresponding points and asymptote for the transformed function, $g(x) = a \cdot e^{x-h} + k$.

	$f(x) = e^x$	$g(x) = a \cdot e^{x-h} + k$
First reference point	$(0, 1)$	$(h, a + k)$
Second reference point	$(1, e)$	$(h + 1, ae + k)$
Asymptote	$y = 0$	$y = k$

Example 1 **Given a function of the form $g(x) = a \cdot e^{x-h} + k$, identify the reference points and use them to draw the graph. State the transformations that compose the combined transformation, the asymptote, the domain, and range. Write the domain and range using set notation.**

A $g(x) = 3 \cdot e^{x+1} + 4$

Compare $g(x) = 3 \cdot e^{x+1} + 4$ to the general form $g(x) = a \cdot e^{x-h} + k$ to find that $h = -1$, $k = 4$, and $a = 3$.

Find the reference points of $f(x) = 3 \cdot e^{x+1} + 4$.

$(0, 1) \rightarrow (h, a + k) = (-1, 3 + 4) = (-1, 7)$

$(1, e) \rightarrow (h + 1, ae + k) = (-1 + 1, 3e + 4) = (0, 3e + 4)$

State the transformations that compose the combined transformation.

$h = -1$, so the graph is translated 1 unit to the left.

$k = 4$, so the graph is translated 4 units up.

$a = 3$, so the graph is vertically stretched by a factor of 3.

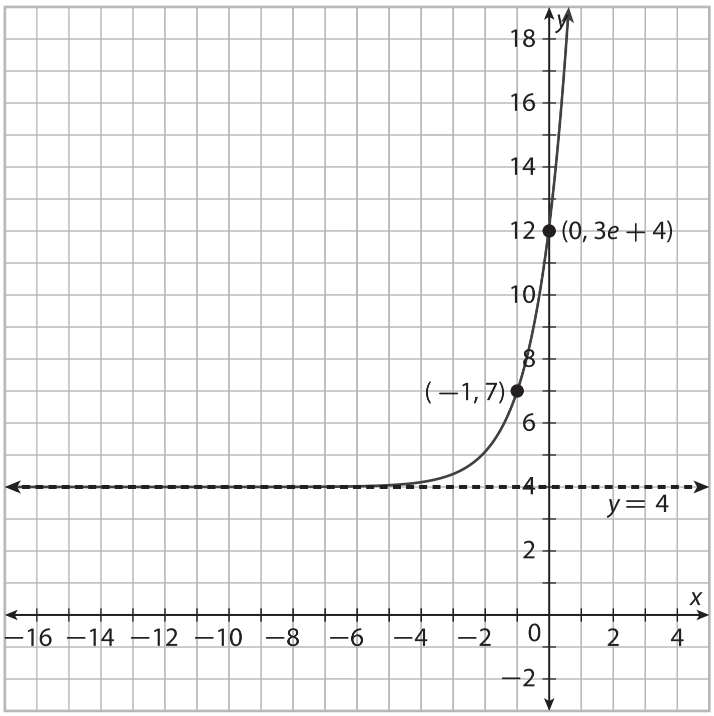
a is positive, so the graph is not reflected across the x-axis.

The asymptote is vertically shifted to $y = k$, so $y = 4$.

The domain is $\left\{x \mid -\infty < x < \infty\right\}$.

The range is $\left\{y \mid y > 4\right\}$.

Use the information to graph the function $g(x) = 3 \cdot e^{x+1} + 4$.



(B) $g(x) = -0.5 \cdot e^{x-2} - 1$

Compare $g(x) = -0.5 \cdot e^{x-2} - 1$ to the general form $g(x) = a \cdot e^{x-h} + k$ to find that $h = \boxed{}$,

$k = \boxed{}$, and $a = \boxed{}$.

Find the reference points of $g(x) = -0.5 \cdot e^{x-2} - 1$.

$$(0, 1) \rightarrow (h, a + k) = \left(\boxed{}, \boxed{} + \boxed{}\right) = \left(\boxed{}, \boxed{}\right)$$

$$(1, e) \rightarrow (h + 1, ae + k) = \left(\boxed{} + 1, \boxed{}e + \boxed{}\right) = \left(\boxed{}, \boxed{}\right)$$

State the transformations that compose the combined transformation.

$h = \boxed{}$, so the graph is translated $\boxed{}$ units to the ________.

$k = \boxed{}$, so the graph is translated $\boxed{}$ unit ________.

$a = \boxed{}$, so the graph is vertically ____________

by a factor of $\boxed{}$.

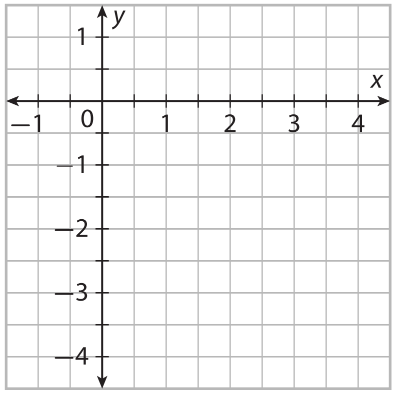
a is negative, so the graph is reflected across the $\boxed{}$-axis.

The asymptote is vertically shifted to $y = k$, so $y = \boxed{}$.

The domain is $\left\{x \middle| \boxed{}\right\}$.

The range is $\left\{y \middle| \boxed{}\right\}$.

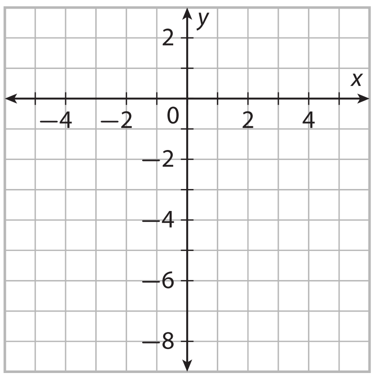
Use the information to graph the function $g(x) = -0.5 \cdot e^{x-2} - 1$.



Your Turn

Given a function of the form $g(x) = a \cdot e^{x-h} + k$, identify the reference points and use them to draw the graph. State the asymptote, domain, and range. Write the domain and range using set notation.

2. $g(x) = (-1) \cdot e^{x+2} - 3$

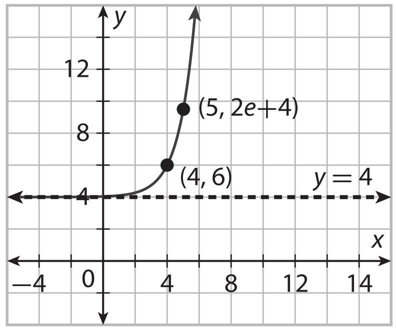


Explain 2 Writing Equations for Combined Transformations of $f(x) = e^x$

If you are given the transformed graph $g(x) = a \cdot e^{x-h} + k$, it is possible to write the equation of the transformed graph by using the reference points $(h, a + k)$ and $(1 + h, ae + k)$.

Example 2 **Write the function whose graph is shown. State the domain and range in set notation.**

(A) First, look at the labeled points on the graph.



$(h, a + k) = (4, 6)$

$(1 + h, ae + k) = (5, 2e + 4)$

Find a, h, and k.

$(h, a + k) = (4, 6)$, so $h = 4$.

$(1 + h, ae + k) = (5, 2e + 4)$, so $ae + k = 2e + 4$.
Therefore, $a = 2$ and $k = 4$.

Write the equation by substituting the values of a, h, and k into the function $g(x) = a \cdot e^{x-h} + k$.

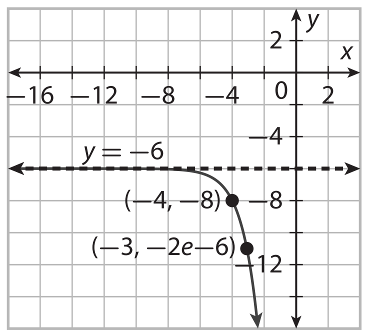
$g(x) = 2e^{x-4} + 4$

State the domain and range.

Domain: $\{x \mid -\infty < x < \infty\}$

Range: $\{y \mid y > 4\}$

(B) First, look at the labeled points on the graph.



$(h, a + k) = \left(\boxed{}, \boxed{}\right)$

$(1 + h, ae + k) = \left(\boxed{}, \boxed{}\right)$

Find a, h, and k.

$(h, a + k) = (-4, -8)$, so $h = \boxed{}$.

$(1 + h, ae + k) = (-3, -2e - 6)$, so $ae + k = \boxed{}$.

Therefore, $a = \boxed{}$ and $k = \boxed{}$.

Write the equation by substituting the values of a, h, and k into the function $g(x) = a \cdot e^{x-h} + k$.

$g(x) = \boxed{}$

State the domain and range.

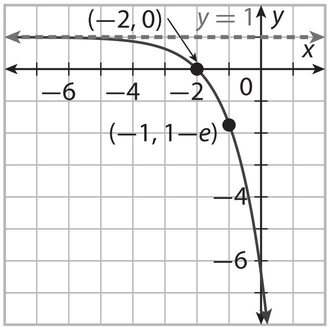
Domain: $\{x \mid \boxed{}\}$

Range: $\{y \mid \boxed{}\}$

Your Turn

Write the function whose graph is shown. State the domain and range in set notation.

3.

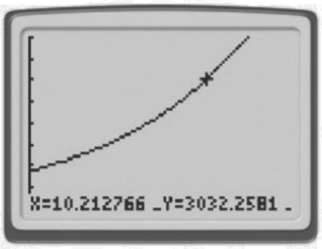


Explain 3 Modeling with Exponential Functions Having Base e

Although the function $f(x) = e^x$ has base $e \approx 2.718$, the function $g(x) = e^{cx}$ can have any positive base (other than 1) by choosing an appropriate positive or negative value of the constant c. This is because you can write $g(x)$ as $(e^c)^x$ by using the Power of a Power Property of Exponents.

Example 3 **Solve each problem using a graphing calculator. Then determine the growth rate or decay rate of the function.**

Ⓐ The Dow Jones index is a stock market index for the New York Stock Exchange. The Dow Jones index for the period 1980-2000 can be modeled by $V_{DJ}(t) = 878e^{0.121t}$, where t is the number of years after 1980. Determine how many years after 1980 the Dow Jones index reached 3000.



Use a graphing calculator to graph the function.

The value of the function is about 3000 when $x \approx 10.2$. So, the Dow Jones index reached 3000 after 10.2 years, or after the year 1990.

In an exponential growth model of the form $f(x) = ae^{cx}$, the growth factor $1 + r$ is equal to e^c.

To find r, first rewrite the function in the form $f(x) = a(e^c)^x$.

$$V_{DJ}(t) = 878e^{0.121t}$$

$$= 878\left(e^{0.121}\right)^t$$

Find r by using $1 + r = e^c$.

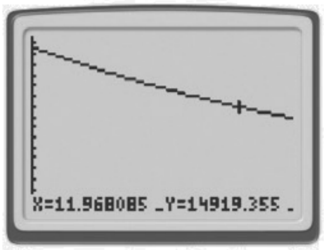
$$1 + r = e^c$$

$$1 + r = e^{0.121}$$

$$r = e^{0.121} - 1 \approx 0.13$$

So, the growth rate is about 13%.

(B) The Nikkei 225 index is a stock market index for the Tokyo Stock Exchange. The Nikkei 225 index for the period 1990-2010 can be modeled by $V_{N225}(t) = 23{,}500e^{-0.0381t}$, where t is the number of years after 1990. Determine how many years after 1990 the Nikkei 225 index reached 15,000.



Use a graphing calculator to graph the function.

The value of the function is about 15,000 when $x \approx$ ____. So, the Nikkei 225 index reached 15,000 after ____ years, or after the year ____.

In an exponential decay model of the form $f(x) = ae^{cx}$, the decay factor ____ is equal to e^c.

To find r, first rewrite the function in the form $f(x) = a(e^c)^x$.

$$V_{N225}(t) = 23{,}500e^{-0.0381t}$$
$$= 23{,}500\left(\boxed{}\right)^t$$

Find r by using $1 - r = e^c$.

$$1 - r = e^c$$
$$1 - r = \boxed{}$$
$$r = \boxed{} \approx \boxed{}$$

So, the decay rate is ____%.

Your Turn

4. A paleontologist uncovers a fossil of a saber-toothed cat in California. The paleontologist analyzes the fossil and concludes that the specimen contains 15% of its original carbon-14. The percent of original carbon-14 in a specimen after t years can be modeled by $N(t) = 100e^{-0.00012t}$, where t is the number of years after the specimen died. Use a graphing calculator to determine the age of the fossil. Then determine the decay rate of the function.

Elaborate

5. Which transformations of $f(x) = e^x$ change the function's end behavior?

6. Which transformations change the location of the graph's y-intercept?

7. Why can the function $f(x) = ae^{cx}$ be used as an exponential growth model and as an exponential decay model? How can you tell if the function represents growth or decay?

8. **Essential Question Check-In** How are reference points helpful when graphing transformations of $f(x) = e^x$ or when writing equations for transformed graphs?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. What does the value of $f(x) = \left(1 + \frac{1}{x}\right)^x$ approach as x increases without bound?

2. Identify the key attributes of $f(x) = e^x$, including the domain and range in set notation, the end behavior, and all intercepts.

Predict the effect of the parameters h, k, or a on the graph of the parent function $f(x) = e^x$. Identify any changes of domain, range, or end behavior.

3. $g(x) = f\left(x - \frac{1}{2}\right)$

4. $g(x) = f(x) - \frac{5}{2}$

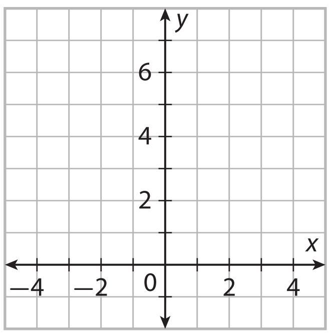
5. $g(x) = -\frac{1}{4} f(x)$

6. $g(x) = \frac{27}{2} f(x)$

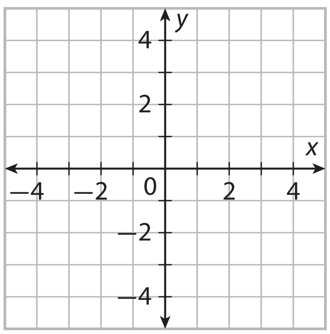
7. The graph of $f(x) = ce^x$ crosses the y-axis at $(0, c)$, where c is some constant. Where does the graph of $g(x) = f(x) - \text{d}$ cross the y-axis?

Given the function of the form $g(x) = a \cdot e^{x-h} + k$, identify the reference points and use them to draw the graph. State the domain and range in set notation.

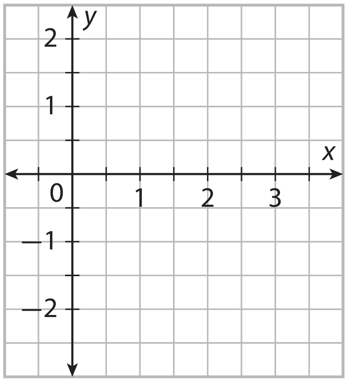
8. $g(x) = e^{x-1} + 2$



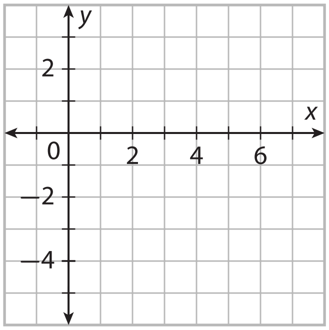
9. $g(x) = -e^{x+1} - 1$



10. $g(x) = \frac{3}{2}e^{x-1} - 3$

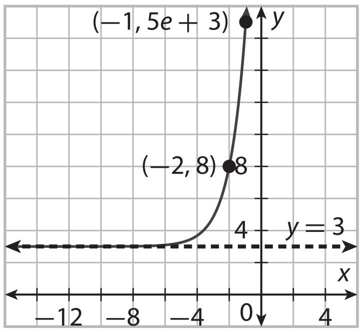


11. $g(x) = -\frac{5}{3}e^{x-4} + 2$

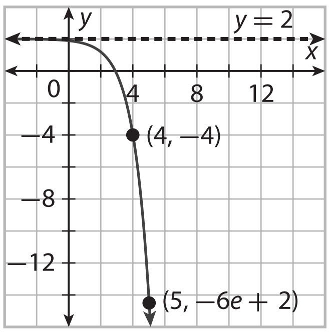


Write the function whose graph is shown. State the domain and range in set notation.

12.



13.



Solve each problem using a graphing calculator. Then determine the growth rate or decay rate of the function.

14. **Medicine** Technetium-99m, a radioisotope used to image the skeleton and the heart muscle, has a half-life of about 6 hours. Use the decay function $N(t) = N_0e^{-0.1155t}$, where N_0 is the initial amount and t is the time in hours, to determine how many hours it takes for a 250 milligram dose to decay to 16 milligrams.

15. **Ecology** The George River herd of caribou in Canada was estimated to be about 4700 in 1954 and grew at an exponential rate to about 472,000 in 1984. Use the exponential growth function $P(t) = P_0e^{0.154t}$, where P_0 is the initial population, t is the time in years after 1954, and $P(t)$ is the population at time t, to determine how many years after 1984 the herd reached 25 million.

H.O.T. Focus on Higher Order Thinking

16. Explain the Error A classmate claims that the function $g(x) = -4e^{x-5} + 6$ is the parent function $f(x) = e^x$ reflected across the y-axis, vertically compressed by a factor of 4, translated to the left 5 units, and translated up 6 units. Explain what the classmate described incorrectly and describe $g(x)$ as a series of transformations of $f(x)$.

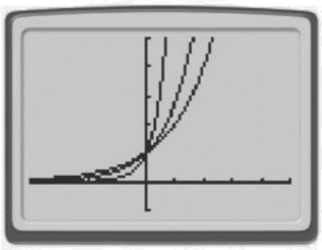
17. Multi-Step Newton's law of cooling states that the temperature of an object decreases exponentially as a function of time, according to $T = T_s + \left(T_0 - T_s\right)e^{-kt}$, where T_0 is the initial temperature of the liquid, T_s is the surrounding temperature, and k is a constant. For a time in minutes, the constant for coffee is approximately 0.283. The corner coffee shop has an air temperature of 70°F and serves coffee at 206°F. Coffee experts say coffee tastes best at 140°F.

a. How long does it take for the coffee to reach its best temperature?

b. The air temperature on the patio outside the coffee shop is 86 °F. How long does it take for coffee to reach its best temperature there?

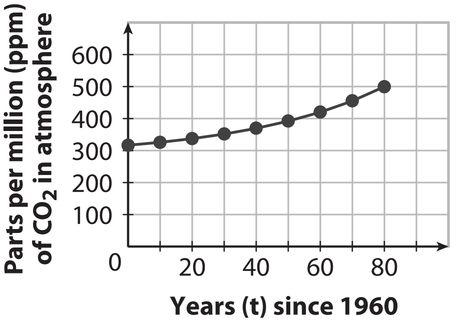
c. Find the time it takes for the coffee to cool to 71°F in both the coffee shop and the patio. Explain how you found your answer.

18. Analyze Relationships The graphing calculator screen shows the graphs of the functions $f(x) = 2^x$, $f(x) = 10^x$, and $f(x) = e^x$ on the same coordinate grid. Identify the common attributes and common point(s) of the three graphs. Explain why the point(s) is(are) common to all three graphs.



Lesson Performance Task

The ever-increasing amount of carbon dioxide in Earth's atmosphere is an area of concern for many scientists. In order to more accurately predict what the future consequences of this could be, scientists make mathematic models to extrapolate past increases into the future. A model developed to predict the annual mean carbon dioxide level L in Earth's atmosphere in parts per million t years after 1960 is $L(t) = 36.9 \cdot e^{0.0223t} + 280$.



a. Use the function $L(t)$ to describe the graph of $L(t)$ as a series of transformations of $f(t) = e^t$.

b. Find and interpret $L(80)$, the carbon dioxide level predicted for the year 2040. How does it compare to the carbon dioxide level in 2015?

c. Can $L(t)$ be used as a model for all positive values of t? Explain.