Name ______________________ Class ______________ Date ________

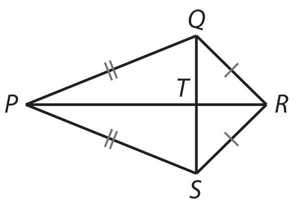
9.5 Properties and Conditions for Kites and Trapezoids

Essential Question: What are the properties of kites and trapezoids?

Resource Locker

Explore Exploring Properties of Kites

A **kite** is a quadrilateral with two distinct pairs of congruent consecutive sides. In the figure, $\overline{PQ} \cong \overline{PS}$, and $\overline{QR} \cong \overline{SR}$, but $\overline{QR} \ncong \overline{QP}$.



Measure the angles made by the sides and diagonals of a kite, noticing any relationships.

(A) Use a protractor to measure $\angle PTQ$ and $\angle QTR$ in the figure. What do your results tell you about the kite's diagonals, $\overline{PR}$ and $\overline{QS}$?

____

(B) Use a protractor to measure $\angle PQR$ and $\angle PSR$ in the figure. How are these opposite angles related?

____

(C) Measure $\angle QPS$ and $\angle QRS$ in the figure. What do you notice?

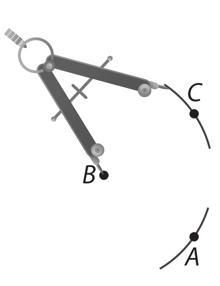
____

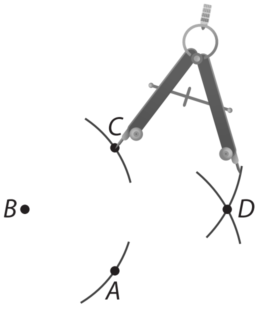
(D) Use a compass to construct your own kite figure on a separate sheet of paper. Begin by choosing a point B. Then use your compass to choose points A and C so that $AB = BC$.

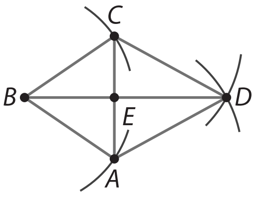
(E) Now change the compass length and draw arcs from both points A and C. Label the intersection of the arcs as point D.

(F) Finally, draw the sides and diagonals of the kite.

Mark the intersection of the diagonals as point E.







(G) Measure the angles of the kite *ABCD* you constructed in Steps D–F and the measure of the angles formed by the diagonals. Are your results the same as for the kite *PQRS* you used in Steps A–C?

Reflect

1. In the kite *ABCD* you constructed in Steps D–F, look at $\angle CDE$ and $\angle ADE$. What do you notice? Is this true for $\angle CBE$ and $\angle ABE$ as well? How can you state this in terms of diagonal $\overline{AC}$ and the pair of non-congruent opposite angles $\angle CBA$ and $\angle CDA$?

2. In the kite *ABCD* you constructed in Steps D–F, look at $\overline{EC}$ and $\overline{EA}$. What do you notice? Is this true for $\overline{EB}$ and $\overline{ED}$ as well? Which diagonal is a perpendicular bisector?

Explain 1 Using Relationships in Kites

The results of the Explore can be stated as theorems.

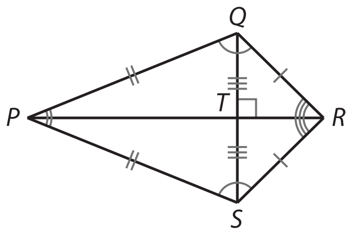
Four Kite Theorems

If a quadrilateral is a kite, then its diagonals are perpendicular.

If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

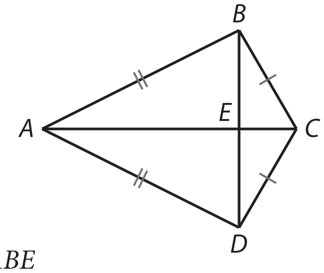
If a quadrilateral is a kite, then one of the diagonals bisects the pair of non-congruent angles.

If a quadrilateral is a kite, then exactly one diagonal bisects the other.



You can use the properties of kites to find unknown angle measures.

Example 1 **In kite *ABCD*, $m\angle BAE = 32°$ and $m\angle BCE = 62°$. Find each measure.**



(A) $m\angle CBE$

Use angle relationships in $\triangle BCE$.

Use the property that the diagonals of a kite are perpendicular, so $m\angle BEC = 90°$.

$\triangle BCE$ is a right triangle.

Therefore, its acute angles are complementary.

$m\angle BCE + m\angle CBE = 90°$

Substitute 62° for $m\angle BCE$, then solve for $m\angle CBE$.

$62° + m\angle CBE = 90°$

$m\angle CBE = 28°$

(B) $m\angle ABE$

$\triangle ABE$ is also a right triangle.

Therefore, its acute angles are complementary.

$m\angle ABE + m\angle \square = \square°$

Substitute 32° for $m\angle \square$, then solve for $m\angle ABE$.

$m\angle ABE + \square° = \square°$

$m\angle ABE = \square°$

Reflect

3. From Part A and Part B, what strategy could you use to determine $m\angle ADC$?

Your Turn

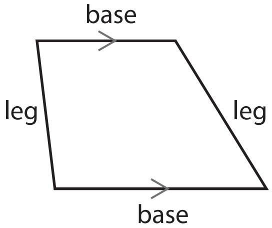
4. Determine $m\angle ADC$ in kite *ABCD*.

Explain 2 Proving that Base Angles of Isosceles Trapezoids Are Congruent

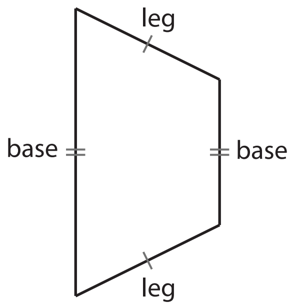
A **trapezoid** is a quadrilateral with at least one pair of parallel sides. The pair of parallel sides of the trapezoid (or either pair of parallel sides if the trapezoid is a parallelogram) are called the *bases* of the trapezoid. The other two sides are called the *legs* of the trapezoid.

A trapezoid has two pairs of *base angles*: each pair consists of the two angles adjacent to one of the bases. An **isosceles trapezoid** is one in which the legs are congruent but not parallel.

Trapezoid



Isosceles trapezoid



Three Isosceles Trapezoid Theorems

If a quadrilateral is an isosceles trapezoid, then each pair of base angles are congruent.

If a trapezoid has one pair of congruent base angles, then the trapezoid is isosceles.

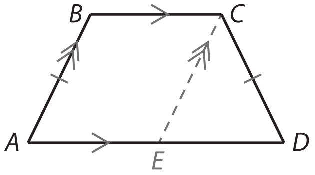
A trapezoid is isosceles if and only if its diagonals are congruent.

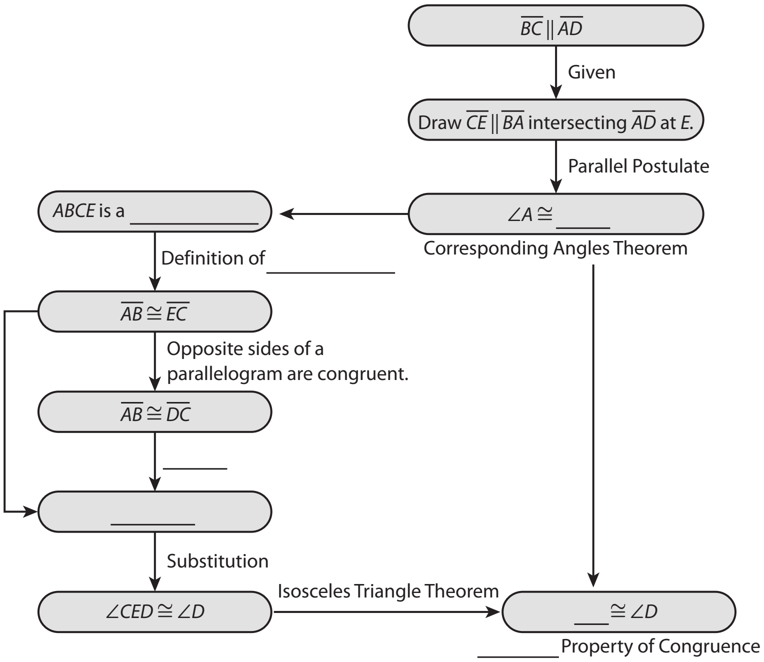
You can use auxiliary segments to prove these theorems.

Example 2 **Complete the flow proof of the first Isosceles Trapezoid Theorem.**

Given: $ABCD$ is an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$, $\overline{AB} \cong \overline{DC}$.

Prove: $\angle A \cong \angle D$





Reflect

5. Explain how the auxiliary segment was useful in the proof.

6. The flow proof in Example 2 only shows that one pair of base angles is congruent. Write a plan for proof for using parallel lines to show that the other pair of base angles $(\angle B$ and $\angle C)$ are also congruent.

____

____

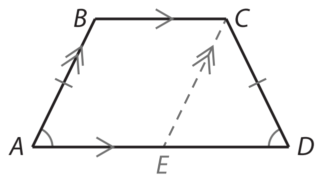
____

Your Turn

7. Complete the proof of the second Isosceles Trapezoid Theorem: If a trapezoid has one pair of base angles congruent, then the trapezoid is isosceles.

Given: $ABCD$ is a trapezoid with $\overline{BC} \parallel \overline{AD}$, $\angle A \cong \angle D$.

Prove: $ABCD$ is an isosceles trapezoid.



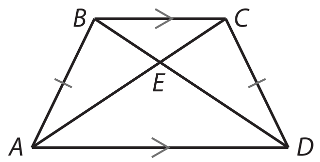
It is given that ________. By the ________________, $\overline{CE}$ can be drawn parallel to _____ so that _____ intersects $\overline{AD}$ at E. By the Corresponding Angles Theorem, $\angle A \cong$ ______. It is given that $\angle A \cong$ _____, so by substitution, ____________. By the Converse of the Isosceles Triangle Theorem, $\overline{CE} \cong$ _____. By definition, _______ is a parallelogram. In a parallelogram, ______________ are congruent, so $\overline{AB} \cong$ _____. By the Transitive Property. of Congruence, $\overline{AB} \cong$ _____. Therefore, by definition, _______ is an ___________________.

Explain 3 Using Theorems about Isosceles Trapezoids

You can use properties of isosceles trapezoids to find unknown values.

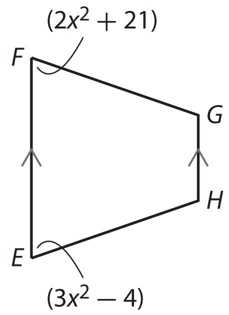
Example 3 **Find each measure or value.**

(A) A railroad bridge has side sections that show isosceles trapezoids. The figure $ABCD$ represents one of these sections. $AC = 13.2$ m and $BE = 8.4$ m. Find DE.



Use the property that the diagonals are congruent.	$\overline{AC} \cong \overline{BD}$
Use the definition of congruent segments.	$AC = BD$
Substitute 13.2 for AC.	$13.2 = BD$
Use the Segment Addition Postulate.	$BE + DE = BD$
Substitute 8.4 for BE and 13.2 for BD.	$8.4 + DE = 13.2$
Subtract 8.4 from both sides.	$DE = 4.8$

(B) Find the value of x so that trapezoid $EFGH$ is isosceles.



For $EFGH$ to be isosceles, each pair of base angles are congruent.

In particular, the pair at E and ____ are congruent. $\angle E \cong \angle$______

Use the definition of congruent angles. $m\angle E = m\angle$_____.

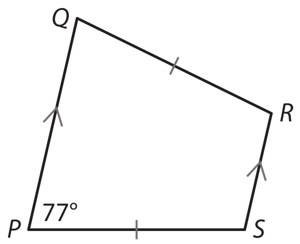
Substitute ________ for $m\angle E$ and __________ for $m\angle$_____. ________ $=$ _________

Substract _____ from both sides and add ____ to both sides. $x^2 =$ _______

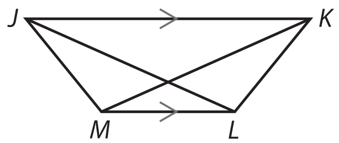
Take the square root of both sides. $x =$ _______ or $x =$ _______

Your Turn

8. In isosceles trapezoid $PQRS$, use the Same-Side Interior Angles Postulate to find $m\angle R$.

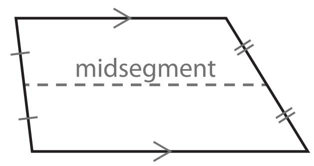


9. $JL = 3y + 6$ and $KM = 22 - y$. Determine the value of y so that trapezoid $JKLM$ is isosceles.



Explain 4 Using the Trapezoid Midsegment Theorem

The **midsegment of a trapezoid** is the segment whose endpoints are the midpoints of the legs.

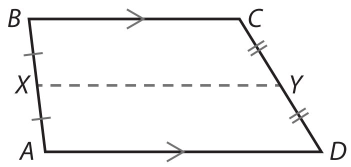


Trapezoid Midsegment Theorem

The midsegment of a trapezoid is parallel to each base, and its length is one half the sum of the lengths of the bases.

$\overline{XY} \parallel \overline{BC}$, $\overline{XY} \parallel \overline{AD}$

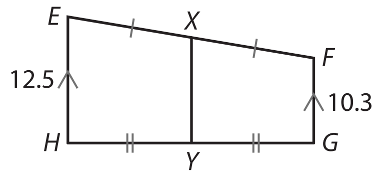
$XY = \frac{1}{2}(BC + AD)$



You can use the Trapezoid Midsegment Theorem to find the length of the midsegment or a base of a trapezoid.

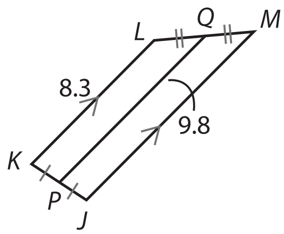
Example 4 **Find each length.**

(A) In trapezoid *EFGH*, find *XY*.



Use the second part of the Trapezoid Midsegment Theorem.	$XY = \frac{1}{2}(EH + FG)$
Substitute 12.5 for *EH* and 10.3 for *FG*.	$= \frac{1}{2}(12.5 + 10.3)$
Simplify.	$= 11.4$

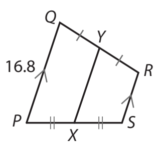
(B) In trapezoid *JKLM*, find *JM*.



Use the second part of the Trapezoid Midsegment Theorem.	$PQ = \frac{1}{2}(\underline{\qquad} + JM)$
Substitute ______ for *PQ* and ______ for ______.	$\underline{\qquad} = \frac{1}{2}(\underline{\qquad} + JM)$
Multiply both sides by 2.	$\underline{\qquad} = \underline{\qquad} + JM$
Subtract ______ from both sides.	$\underline{\qquad} = JM$

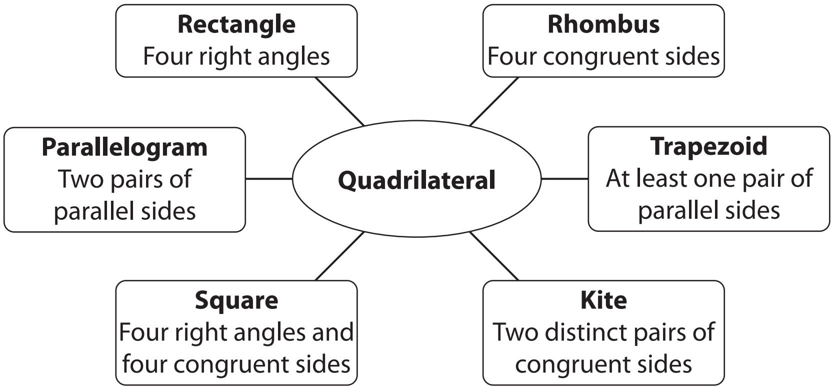
Your Turn

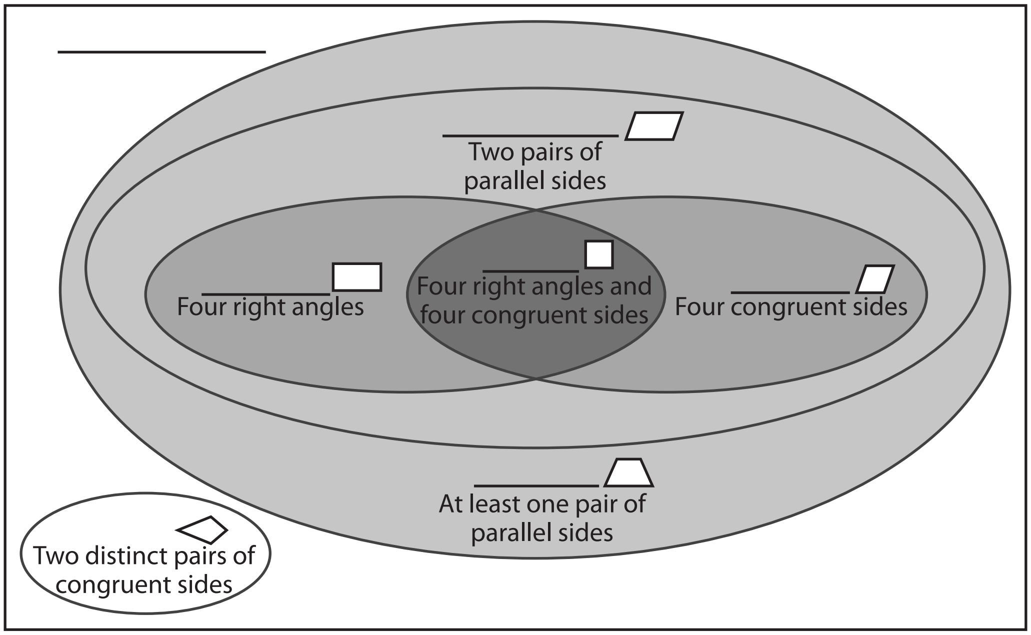
10. In trapezoid *PQRS*, $PQ = 2RS$. Find *XY*.



Elaborate

11. Use the information in the graphic organizer to complete the Venn diagram.





What can you conclude about all parallelograms? ______________________________

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12. Discussion The Isosceles Trapezoid Theorem about congruent diagonals is in the form of a biconditional statement. Is it possible to state the two isosceles trapezoid theorems about base angles as a biconditional statement? Explain.

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13. Essential Question Check-In Do kites and trapezoids have properties that are related to their diagonals? Explain.

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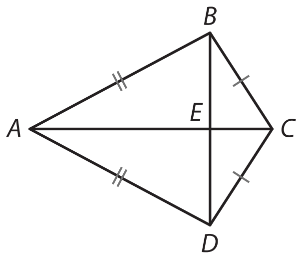
Evaluate: Homework and Practice

In kite $ABCD$, $m\angle BAE = 28°$ and $m\angle BCE = 57°$. Find each measure.



1. $m\angle ABE$

2. $m\angle CBE$

3. $m\angle ABC$

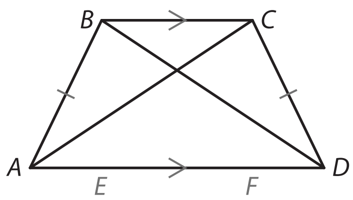
4. $m\angle ADC$

Using the first and second Isosceles Trapezoid Theorems, complete the proofs of each part of the third Isosceles Trapezoid Theorem: *A trapezoid is isosceles if and only if its diagonals are congruent.*

5. Prove part 1: If a trapezoid is isosceles, then its diagonals are congruent.

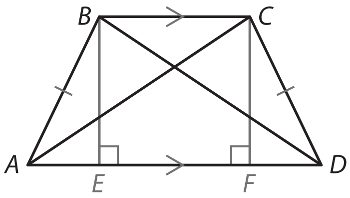
Given: $ABCD$ is an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$, $\overline{AB} \cong \overline{DC}$.

Prove: $\overline{AC} \cong \overline{DB}$



It is given that $\overline{AB} \cong \overline{DC}$. By the first Trapezoid Theorem, $\angle BAD \cong$ ____________, and by the Reflexive Property of Congruence, ____________. By the SAS Triangle Congruence Theorem, $\triangle ABD \cong \triangle DCA$, and by ____________, $\overline{AC} \cong \overline{DB}$.

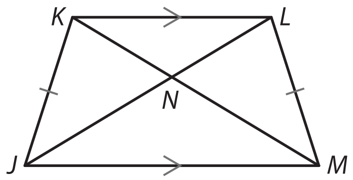
6. Prove part 2: If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.
Given: $ABCD$ is a trapezoid with $\overline{BC} \parallel \overline{AD}$ and diagonals $\overline{AC} \cong \overline{DB}$.
Prove: $ABCD$ is an isosceles trapezoid.



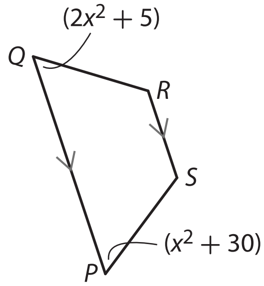
Statements	Reasons
1. Draw $\overline{BE} \perp \overline{AD}$ and $\overline{CF} \perp \overline{AD}$.	**1.** There is only one line through a given point perpendicular to a given line, so each auxiliary line can be drawn.
2. $\overline{BE} \parallel \overline{CF}$	**2.** Two lines perpendicular to the same line are parallel.
3. ________	**3.** Given
4. *BCFE* is a parallelogram.	**4.** ____________________ *(Steps 2, 3)*
5. $\overline{BE} \cong$ ______	**5.** If a quadrilateral is a parallelogram, then its opposite sides are congruent.
6. $\overline{AC} \cong \overline{DB}$	**6.** ______
7. ____________________	**7.** Definition of perpendicular lines
8. $\triangle BED \cong \triangle CFA$	**8.** HL Triangle Congruence Theorem *(Steps 5–7)*
9. $\angle BDE \cong \angle CAF$	**9.** ______
10. $\angle CBD \cong$ ______, ______ $\cong \angle CAF$	**10.** Alternate Interior Angles Theorem
11. $\angle CBD \cong$ ______	**11.** Transitive Property of Congruence *(Steps 9, 10)*
12. ________	**12.** Given
13. $\overline{BC} \cong \overline{BC}$	**13.** ____________________
14. $\triangle ABC \cong \triangle DCB$	**14.** ____________________ *(Steps 12, 13)*
15. $\angle BAC \cong \angle CDB$	**15.** CPCTC
16. $\angle BAD \cong$ ______	**16.** Angle Addition Postulate
17. *ABCD* is isosceles.	**17.** If a trapezoid has one pair of base angles congruent, then the trapezoid is isosceles.

Use the isosceles trapezoid to find each measure or value.

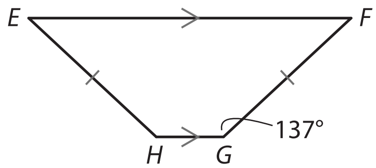
7. $LJ = 19.3$ and $KN = 8.1$. Determine MN.



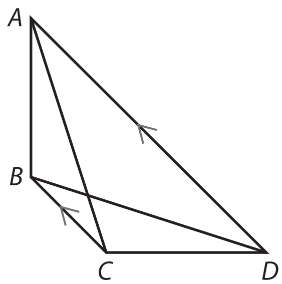
8. Find the positive value of x so that trapezoid $PQRS$ is isosceles.



9. In isosceles trapezoid $EFGH$, use the Same-Side Interior Angles Postulate to determine $m\angle E$.

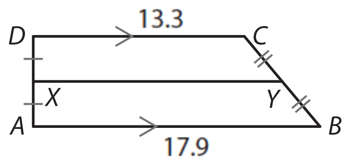


10. $AC = 3y + 12$ and $BD = 27 - 2y$. Determine the value of y so that trapezoid $ABCD$ is isosceles.

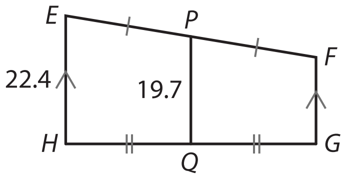


Find the unknown segment lengths in each trapezoid.

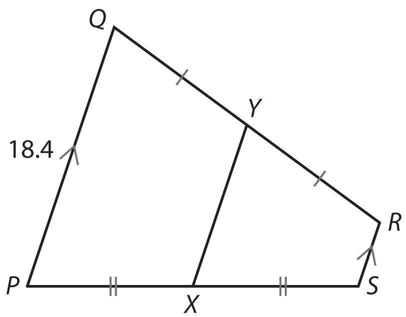
11. In trapezoid $ABCD$, find XY.



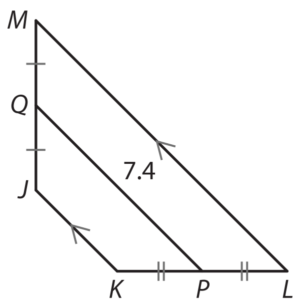
12. In trapezoid $EFGH$, find FG.



13. In trapezoid $PQRS$, $PQ = 4RS$. Determine XY.



14. In trapezoid $JKLM$, $PQ = 2JK$. Determine LM.



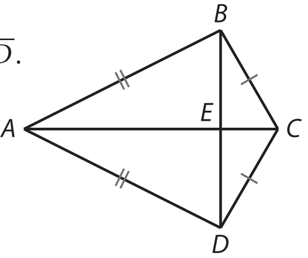
15. Determine whether each of the following describes a kite or a trapezoid. Select the correct answer for each lettered part.

A. Has two distinct pairs of congruent consecutive sides	◯ kite	◯ trapezoid
B. Has diagonals that are perpendicular	◯ kite	◯ trapezoid
C. Has at least one pair of parallel sides	◯ kite	◯ trapezoid
D. Has exactly one pair of opposite angles that are congruent	◯ kite	◯ trapezoid
E. Has two pairs of base angles	◯ kite	◯ trapezoid

16. **Multi-Step** Complete the proof of each of the four Kite Theorems. The proof of each of the four theorems relies on the same initial reasoning, so they are presented here in a single two-column proof.

Given: $ABCD$ is a kite, with $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$.

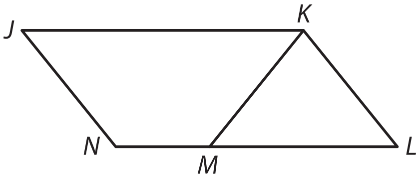
Prove: (i) $\overline{AC} \perp \overline{BD}$;
(ii) $\angle ABC \cong \angle ADC$;
(iii) $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$;
(iv) $\overline{AC}$ bisects $\overline{BD}$.



Statements	Reasons
1. $\overline{AB} \cong \overline{AD}$, $\overline{CB} \cong \overline{CD}$	**1.** Given
2. $\overline{AC} \cong$ ______	**2.** Reflexive Property of Congruence
3. $\triangle ABC \cong \triangle ADC$	**3.** ______________________ *(Steps 1, 2)*
4. $\angle BAE \cong$ ______	**4.** CPCTC
5. $\overline{AE} \cong \overline{AE}$	**5.** Reflexive Property of Congruence
6. ____________	**6.** SAS Triangle Congruence Theorem *(Steps 1, 4, 5)*
7. $\angle AEB \cong \angle AED$	**7.** ______
8. $\overline{AC} \perp \overline{BD}$	**8.** If two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.
9. $\angle ABC \cong$ ______	**9.** ______ *(Step 3)*
10. $\angle BAC \cong$ ______ and ______ $\cong \angle DCA$	**10.** ______ *(Step 3)*
11. $\overline{AC}$ bisects $\angle BAD$ and $\angle BCD$.	**11.** Definition of ____________
12. ______ $\cong$ ______	**12.** CPCTC *(Step 6)*
13. $\overline{AC}$ bisects $\overline{BD}$.	**13.** ______________________

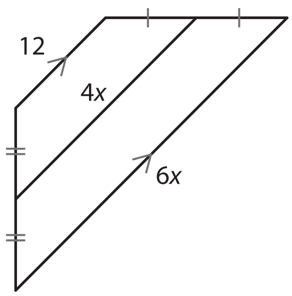
17. Given: *JKLN* is a parallelogram. *JKMN* is an isosceles trapezoid.

Prove: $\triangle KLM$ is an isosceles triangle.

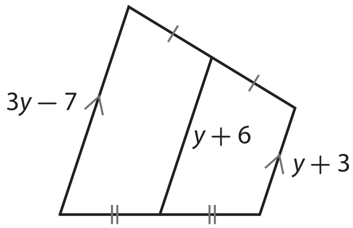


Algebra **Find the length of the midsegment of each trapezoid.**

18.



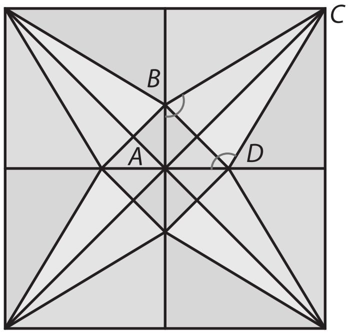
19.



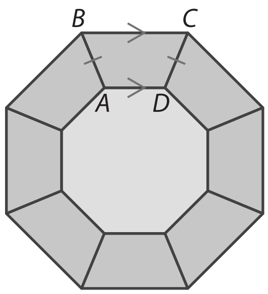
20. **Represent Real-World Problems** A set of shelves fits an attic room with one sloping wall. The left edges of the shelves line up vertically, and the right edges line up along the sloping wall. The shortest shelf is 32 in. long, and the longest is 40 in. long. Given that the three shelves are equally spaced vertically, what total length of shelving is needed?

21. **Represent Real-World Problems** A common early stage in making an origami model is known as the kite. The figure shows a paper model at this stage unfolded.

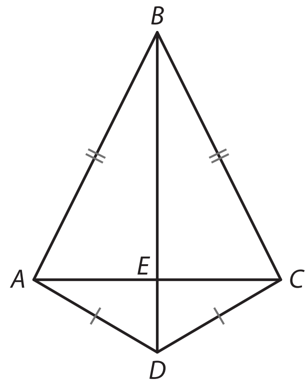
The folds create four geometric kites. Also, the 16 right triangles adjacent to the corners of the paper are all congruent, as are the 8 right triangles adjacent to the center of the paper. Find the measures of all four angles of the kite labeled *ABCD* (the point A is the center point of the diagram). Use the facts that $\angle B \cong \angle D$ and that the interior angle sum of a quadrilateral is 360°.



22. **Analyze Relationships** The window frame is a regular octagon. It is made from eight pieces of wood shaped like congruent isosceles trapezoids. What are $m\angle A$, $m\angle B$, $m\angle C$, and $m\angle D$ in trapezoid $ABCD$?



23. **Explain the Error** In kite $ABCD$, $m\angle BAE = 66°$ and $m\angle ADE = 59°$. Terrence is trying to find $m\angle ABC$. He knows that $\overline{BD}$ bisects $\overline{AC}$, and that therefore $\triangle AED \cong \triangle CED$. He reasons that $\angle ADE \cong \angle CDE$, so that $m\angle ADC = 2(59°) = 118°$, and that $\angle ABC \cong \angle ADC$ because they are opposite angles in the kite, so that $m\angle ABC = 118°$. Explain Terrence's error and describe how to find $m\angle ABC$.



24. Complete the table to classify all quadrilateral types by the rotational symmetries and line symmetries they must have. Identify any patterns that you see and explain what these patterns indicate.

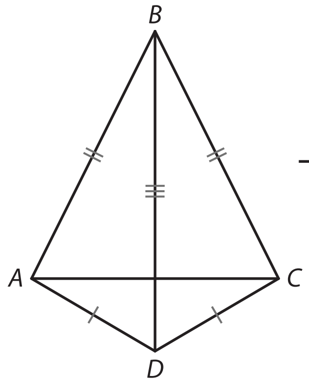
Quadrilateral	Angle of Rotational Symmetry	Number of Line Symmetries
kite		1
non-isosceles trapezoid	none	
isosceles trapezoid		
parallelogram	180°	
rectangle		
rhombus		
square		

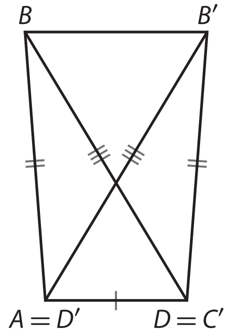
H.O.T. Focus on Higher Order Thinking

25. Communicate Mathematical Ideas Describe the properties that rhombuses and kites have in common, and the properties that are different.

26. Analyze Relationships In kite *ABCD*, triangles *ABD* and *CBD* can be rotated and translated, identifying $\overline{AD}$ with $\overline{CD}$ and joining the remaining pair of vertices, as shown in the figure. Why is this process guaranteed to produce an isosceles trapezoid?

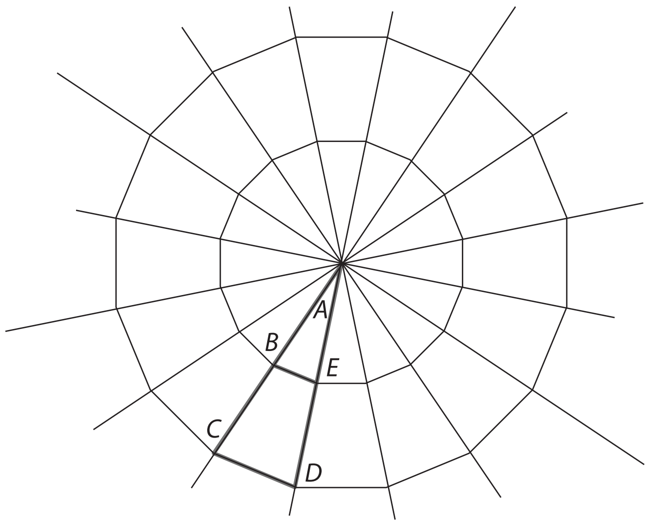
Next, suggest a process guaranteed to produce a kite from an isosceles trapezoid, using figures to illustrate your process.





Lesson Performance Task

This model of a spider web is made using only isosceles triangles and isosceles trapezoids.



a. All of the figures surrounding the center of the web are congruent to figure *ABCDE*. Find m∠*A*. Explain how you found your answer.

b. Find m∠*ABE* and m∠*AEB*.

c. Find m∠*CBE* and m∠*DEB*.

d. Find m∠*C* and m∠*D*.