

# 8.2 Graphing More Complicated Rational Functions



Resource Locker

**Essential Question:** What features of the graph of a rational function should you identify in order to sketch the graph? How do you identify those features?

## Explore 1 Investigating Domains and Vertical Asymptotes of More Complicated Rational Functions

You know that the rational function  $f(x) = \frac{1}{x-2} + 3$  has the domain  $\{x | x \neq 2\}$  because the function is undefined at  $x = 2$ . Its graph has the vertical asymptote  $x = 2$  because as  $x \rightarrow 2^+$  ( $x$  approaches 2 from the right),  $f(x) \rightarrow +\infty$ , and as  $x \rightarrow 2^-$  ( $x$  approaches 2 from the left),  $f(x) \rightarrow -\infty$ . In this Explore, you will investigate the domains and vertical asymptotes of other rational functions.

- A** Complete the table by identifying each function's domain based on the  $x$ -values for which the function is undefined. Write the domain using an inequality, set notation, and interval notation. Then state the equations of what you think the vertical asymptotes of the function's graph are.

Function	Domain	Possible Vertical Asymptotes
$f(x) = \frac{x + 3}{x - 1}$		
$f(x) = \frac{(x + 5)(x - 1)}{x + 1}$		
$f(x) = \frac{x - 4}{(x + 1)(x - 1)}$		
$f(x) = \frac{2x^2 - 3x + 9}{x^2 - x - 6}$		

- B** Using a graphing calculator, graph each of the functions from Step A, and check to see if vertical asymptotes occur where you expect. Are there any unexpected results?

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- C** Examine the behavior of  $f(x) = \frac{x+3}{x-1}$  near  $x = 1$ .

First, complete the tables.

x approaches 1 from the right	
x	$f(x) = \frac{x+3}{x-1}$
1.1	
1.01	
1.001	

x approaches 1 from the left	
x	$f(x) = \frac{x+3}{x-1}$
0.9	
0.99	
0.999	

Next, summarize the results.

- As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow$  .

The behaviour of  $f(x) = \frac{x+3}{x-1}$  near  $x = 1$  shows that the graph of  $f(x)$  [has/does not] have a vertical asymptote at  $x = 1$ .

- D** Examine the behavior of  $f(x) = \frac{(x+5)(x-1)}{(x+1)}$  near  $x = -1$ .

First, complete the tables.

x approaches -1 from the right	
x	$f(x) = \frac{(x+5)(x-1)}{x+1}$
-0.9	
-0.99	
-0.999	

x approaches -1 from the left	
x	$f(x) = \frac{(x+5)(x-1)}{x+1}$
-1.1	
-1.01	
-1.001	

Next, summarize the results.

- As  $x \rightarrow -1^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow -1^-$ ,  $f(x) \rightarrow$  .

The behavior of  $f(x) = \frac{(x+5)(x-1)}{(x+1)}$  near  $x = -1$  shows that the graph of  $f(x)$  [has/does not have] a vertical asymptote at  $x = -1$ .

- E Examine the behavior of  $f(x) = \frac{x - 4}{(x + 1)(x - 1)}$  near  $x = -1$  and  $x = 1$ .

First, complete the tables. Round results to the nearest tenth.

x approaches -1 from the right	
x	$f(x) = \frac{x - 4}{(x + 1)(x - 1)}$
-0.9	
-0.99	
-0.999	

x approaches -1 from the left	
x	$f(x) = \frac{x - 4}{(x + 1)(x - 1)}$
-1.1	
-1.01	
-1.001	

x approaches 1 from the right	
x	$f(x) = \frac{x - 4}{(x + 1)(x - 1)}$
1.1	
1.01	
1.001	

x approaches 1 from the left	
x	$f(x) = \frac{x - 4}{(x + 1)(x - 1)}$
0.9	
0.99	
0.999	

Next, summarize the results.

- As  $x \rightarrow -1^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow -1^-$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow$  .

The behavior of  $f(x) = \frac{x - 4}{(x + 1)(x - 1)}$  near  $x = -1$  shows that the graph of  $f(x)$  has/does not have a vertical asymptote at  $x = -1$ . The behavior of  $f(x) = \frac{x - 4}{(x + 1)(x - 1)}$  near  $x = 1$  shows that the graph of  $f(x)$  has/does not have a vertical asymptote at  $x = 1$ .

- F Examine the behavior of  $f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$  near  $x = -2$  and  $x = 3$ .

First, complete the tables. Round results to the nearest ten thousandth if necessary.

x approaches -2 from the right	
x	$f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$
-1.9	
-1.99	
-1.999	

x approaches -2 from the left	
x	$f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$
-2.1	
-2.01	
-2.001	

x approaches 3 from the right	
x	$f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$
3.1	
3.01	
3.001	

x approaches 3 from the left	
x	$f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$
2.9	
2.99	
2.999	

Next, summarize the results.

- As  $x \rightarrow -2^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow -2^-$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow 3^+$ ,  $f(x) \rightarrow$  .
- As  $x \rightarrow 3^-$ ,  $f(x) \rightarrow$  .

The behavior of  $f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$  near  $x = -2$  shows that the graph of  $f(x)$  has/does not have a vertical asymptote at  $x = -2$ . The behavior of  $f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$  near  $x = 3$  shows that the graph of  $f(x)$  has/does not have a vertical asymptote at  $x = 3$ .

### Reflect

- Rewrite  $f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$  so that its numerator and denominator are factored. How does this form of the function explain the behavior of the function near  $x = 3$ ?

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- Discussion** When you graph  $f(x) = \frac{2x^2 - 3x - 9}{x^2 - x - 6}$  on a graphing calculator, you can't tell that the function is undefined for  $x = 3$ . How does using the calculator's table feature help? What do you think the graph should look like to make it clear that the function is undefined at  $x = 3$ ?

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## Explain 1 Sketching the Graphs of More Complicated Rational Functions

As you have seen, there can be breaks in the graph of a rational function. These breaks are called *discontinuities*, and there are two kinds:

1. When a rational function has a factor in the denominator that is not also in the numerator, an *infinite discontinuity* occurs at the value of  $x$  for which the factor equals 0. On the graph of the function, an infinite discontinuity appears as a vertical asymptote.
2. When a rational function has a factor in the denominator that is also in the numerator, a *point discontinuity* occurs at the value of  $x$  for which the factor equals 0. On the graph of the function, a point discontinuity appears as a “hole.”

The graph of a rational function can also have a horizontal asymptote, or even an asymptote that is a line that is neither horizontal nor vertical. This is determined by the degrees and leading coefficients of the function’s numerator and denominator. Examine the following rational expressions, which include polynomial quotients rewritten using long division as a quotient plus a remainder that approaches 0 as  $x$  increases or decreases without bound.

$$\frac{1}{x-1} = 0 \text{ as } x \Rightarrow \pm\infty$$

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1} \quad (x \neq -1), \text{ so } \frac{x+1}{x^2-1} \Rightarrow 0 \text{ as } x \Rightarrow \pm\infty$$

$$\frac{x+3}{x-1} = 1 + \frac{4}{x-1} \Rightarrow 1 \text{ as } x \Rightarrow \pm\infty$$

$$\frac{4x+3}{x-1} = 4 + \frac{7}{x-1} \Rightarrow 4 \text{ as } x \Rightarrow \pm\infty$$

$$\frac{3x^2+x-1}{x-1} = 3x+4 + \frac{3}{x-1} \Rightarrow 3x+4 \text{ as } x \Rightarrow \pm\infty$$

In general, if the numerator is a polynomial  $p(x)$  in standard form with leading coefficient  $a$  and the denominator is a polynomial  $q(x)$  in standard form with leading coefficient  $b$ , then an examination of the function’s end behavior gives the following results.

Relationship between Degree of $p(x)$ and Degree of $q(x)$	Equation of Horizontal Asymptote (if one exists)
Degree of $p(x) <$ degree of $q(x)$	$y = 0$
Degree of $p(x) =$ degree of $q(x)$	$y = \frac{a}{b}$
Degree of $p(x) >$ degree of $q(x)$	There is no horizontal asymptote. The function instead increases or decreases without bound as $x$ increases or decreases without bound. In particular, when the degree of the numerator is 1 more than the degree of the denominator, the function’s graph approaches a slanted line, called a <i>slant asymptote</i> , as $x$ increases or decreases without bound.

You can sketch the graph of a rational function by identifying where vertical asymptotes, “holes,” horizontal asymptotes, and slant asymptotes occur. Using the factors of the numerator and denominator, you can also establish intervals on the  $x$ -axis where either an  $x$ -intercept or a discontinuity occurs and then check the signs of the factors on those intervals to determine whether the graph lies above or below the  $x$ -axis.

**Example 1** Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.)

A  $f(x) = \frac{x + 1}{x - 2}$

Identify vertical asymptotes and "holes."

The function is undefined when  $x - 2 = 0$ , or  $x = 2$ . Since  $x - 2$  is not a factor of the numerator, there is a vertical asymptote rather than a "hole" at  $x = 2$ .

Identify horizontal asymptotes and slant asymptotes.

The numerator and denominator have the same degree and the leading coefficient of each is 1, so there is a horizontal asymptote at  $y = \frac{1}{1} = 1$ .

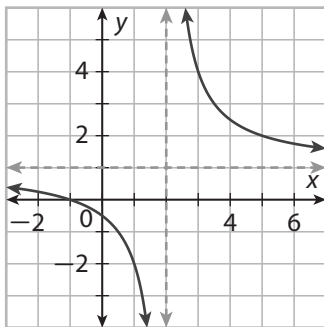
Identify  $x$ -intercepts.

An  $x$ -intercept occurs when  $x + 1 = 0$ , or  $x = -1$ .

Check the sign of the function on the intervals  $x < -1$ ,  $-1 < x < 2$ , and  $x > 2$ .

Interval	Sign of $x + 1$	Sign of $x - 2$	Sign of $f(x) = \frac{x + 1}{x - 2}$
$x < -1$	-	-	+
$-1 < x < 2$	+	-	-
$x > 2$	+	+	+

Sketch the graph using all this information. Then state the domain and range.



Domain:

Inequality:  $x < 2$  or  $x > 2$

Set notation:  $\{x | x \neq 2\}$

Interval notation:  $(-\infty, 2) \cup (2, +\infty)$

Range:

Inequality:  $y < 1$  or  $y > 1$

Set notation:  $\{y | y \neq 1\}$

Interval notation:  $(-\infty, 1) \cup (1, +\infty)$

**B**  $f(x) = \frac{x^2 + x - 2}{x + 3}$

Factor the function's numerator.

$$f(x) = \frac{x^2 + x - 2}{x + 3} = \frac{(x - 1)(x + 2)}{x + 3}$$

Identify vertical asymptotes and "holes."

The function is undefined when  $x + 3 = 0$ , or  $x = \square$ . Since  $x + 3$  is **not** a factor of the numerator, there is a vertical asymptote rather than a "hole" at  $x = \square$ .

Identify horizontal asymptotes and slant asymptotes.

Because the degree of the numerator is 1 more than the degree of the denominator, there is no horizontal asymptote, but there is a slant asymptote. Divide the numerator by the denominator to identify the slant asymptote.

$$\begin{array}{r} x - \square \\ x + 3 \overline{) x^2 + x - 2} \\ \underline{x^2 + 3x} \phantom{- 2} \\ -2x - 2 \\ \underline{-2x - 6} \\ 4 \end{array}$$

So, the line  $y = x - \square$  is the slant asymptote.

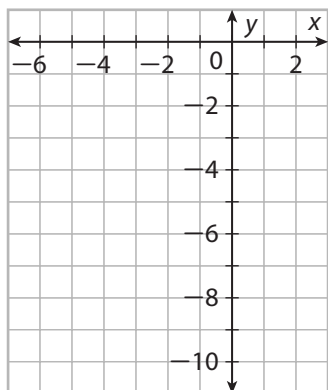
Identify  $x$ -intercepts.

There are two  $x$ -intercepts: when  $x - 1 = 0$ , or  $x = \square$ , and when  $x + 2 = 0$ , or  $x = \square$ .

Check the sign of the function on the intervals  $x < -3$ ,  $-3 < x < -2$ ,  $-2 < x < 1$ , and  $x > 1$ .

Interval	Sign of $x + 3$	Sign of $x + 2$	Sign of $x - 1$	Sign of $f(x) = \frac{(x - 1)(x + 2)}{x + 3}$
$x < -3$	-	-	-	-
$-3 < x < -2$				
$-2 < x < 1$				
$x > 1$				

Sketch the graph using all this information. Then state the domain and range.



Domain:

Inequality:  $x < \square$  or  $x > \square$

Set notation:  $\{x | x \neq \square\}$

Interval notation:  $(-\infty, \square) \cup (\square, +\infty)$

The sketch indicates that the function has a maximum value and a minimum value. Using **3:minimum** from the **CALC** menu on a graphing calculator gives  $-1$  as the minimum value. Using **4:maximum** from the **CALC** menu on a graphing calculator gives  $-5$  as the maximum value.

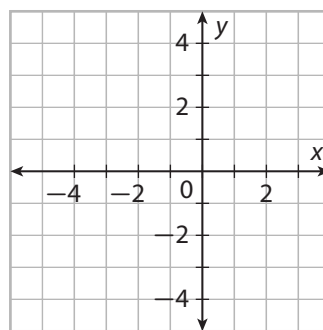
Range: Inequality:  $y < \square$  or  $y > \square$

Set notation:  $\{y \mid y < \square \text{ or } y > \square\}$  Interval notation:  $(-\infty, \square) \cup (\square, +\infty)$

### Your Turn

Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.)

3.  $f(x) = \frac{x + 1}{x^2 + 3x - 4}$



## Explain 2 Modeling with More Complicated Rational Functions

When two real-world variable quantities are compared using a ratio or rate, the comparison is a rational function. You can solve problems about the ratio or rate by graphing the rational function.

**Example 2** Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.

- A** A baseball team has won 32 games out of 56 games played, for a winning percentage of  $\frac{32}{56} \approx 0.571$ . How many consecutive games must the team win to raise its winning percentage to 0.600?

Let  $w$  be the number of consecutive games to be won. Then the total number of games won is the function  $T_{\text{won}}(w) = 32 + w$ , and the total number of games played is the function  $T_{\text{played}}(w) = 56 + w$ .



The rational function that gives the team's winning percentage  $p$  (as a decimal) is

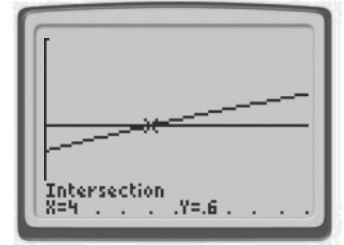
$$p(w) = \frac{T_{\text{won}}(w)}{T_{\text{played}}(w)} = \frac{32 + w}{56 + w}.$$

The domain of the rational function is  $\{w | w \geq 0 \text{ and } w \text{ is a whole number}\}$ . Note that you do not need to exclude  $-56$  from the domain, because only nonnegative whole-number values of  $w$  make sense in this situation.

Since the function models what happens to the team's winning percentage from consecutive wins (no losses), the values of  $p(w)$  start at 0.571 and approach 1 as  $w$  increases without bound. So, the range is

$$\{p | 0.571 \leq p < 1\}.$$

Graph  $y = \frac{32 + x}{56 + x}$  on a graphing calculator using a viewing window that shows 0 to 10 on the  $x$ -axis and 0.5 to 0.7 on the  $y$ -axis. Also graph the line  $y = 0.6$ . To find where the graphs intersect, select **5: intersect** from the **CALC** menu.



So, the team's winning percentage (as a decimal) will be 0.600 if the team wins 4 consecutive games.

- B** Two friends decide spend an afternoon canoeing on a river. They travel 4 miles upstream and 6 miles downstream. In still water, they know that their average paddling speed is 5 miles per hour. If their canoe trip takes 4 hours, what is the average speed of the river's current? To answer the question, use the rational function  $t(c) = \frac{4}{5-c} + \frac{6}{5+c} = \frac{50-2c}{(5-c)(5+c)}$  where  $c$  is the average speed of the current (in miles per hour) and  $t$  is the time (in hours) spent canoeing 4 miles against the current at a rate of  $5 - c$  miles per hour and 6 miles with the current at a rate of  $5 + c$  miles per hour.



In order for the friends to travel upstream, the speed of the current must be less than

their average paddling speed, so the domain of the function is  $\{c | 0 \leq c < \square\}$ . If the friends canoed in still water ( $c = 0$ ), the trip would take a total of  $\frac{4}{5} + \frac{6}{5} = \square$  hours.

As  $c$  approaches 5 from the left, the value of  $\frac{6}{5+c}$  approaches  $\frac{6}{10} = 0.6$  hour, but the value of  $\frac{4}{5-c}$  \_\_\_\_\_. So, the range of the function is  $\{t | t \geq \square\}$ .

Graph  $y = \frac{50 - 2x}{(5 - x)(5 + x)}$  on a graphing calculator using a viewing window that shows 0 to 5 on the  $x$ -axis and 2 to 5 on the  $y$ -axis. Also graph the line  $y = \square$ . To find where the graphs intersect, select **5:intersect** from the **CALC** menu. The calculator shows that the average speed of the current is about \_\_\_\_\_ miles per hour.

**Your Turn**

Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.

4. A saline solution is a mixture of salt and water. A  $p\%$  saline solution contains  $p\%$  salt and  $(100 - p)\%$  water by mass. A chemist has 300 grams of a 4% saline solution that needs to be strengthened to a 6% solution by adding salt. How much salt should the chemist add?

 **Elaborate**

5. How can you show that the vertical line  $x = c$ , where  $c$  is a constant, is an asymptote for the graph of a rational function?

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6. How can you determine the end behavior of a rational function?

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7. **Essential Question Check-In** How do you identify any vertical asymptotes and “holes” that the graph of a rational function has?

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## Evaluate: Homework and Practice



- Online Homework
- Hints and Help
- Extra Practice

State the domain using an inequality, set notation, and interval notation. For any  $x$ -value excluded from the domain, state whether the graph has a vertical asymptote or a “hole” at that  $x$ -value. Use a graphing calculator to check your answer.

1.  $f(x) = \frac{x+5}{x+1}$

2.  $f(x) = \frac{x^2 + 2x - 3}{x^2 - 4x + 3}$

Divide the numerator by the denominator to write the function in the form  $f(x) = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$  and determine the function’s end behavior. Then, using a graphing calculator to examine the function’s graph, state the range using an inequality, set notation, and interval notation.

3.  $f(x) = \frac{3x+1}{x-2}$

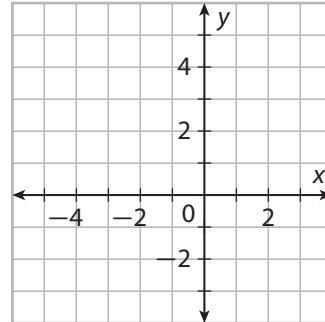
4.  $f(x) = \frac{x}{(x-2)(x+3)}$

5.  $f(x) = \frac{x^2 - 5x + 6}{x-1}$

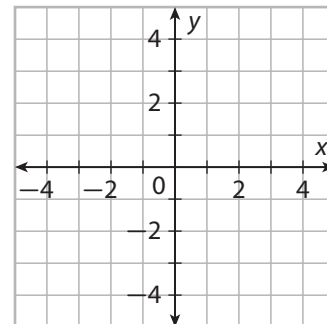
6.  $f(x) = \frac{4x^2 - 1}{x^2 + x - 2}$

Sketch the graph of the given rational function. (If the degree of the numerator is 1 more than the degree of the denominator, find the graph's slant asymptote by dividing the numerator by the denominator.) Also state the function's domain and range using inequalities, set notation, and interval notation. (If your sketch indicates that the function has maximum or minimum values, use a graphing calculator to find those values to the nearest hundredth when determining the range.)

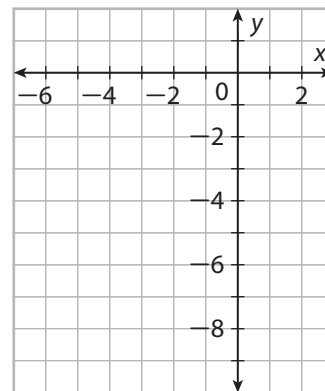
7.  $f(x) = \frac{x-1}{x+1}$



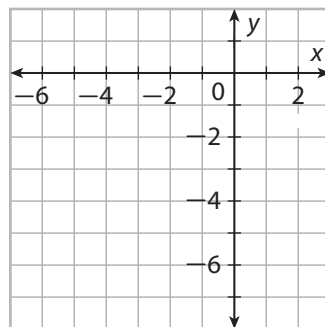
8.  $f(x) = \frac{x-1}{(x-2)(x+3)}$



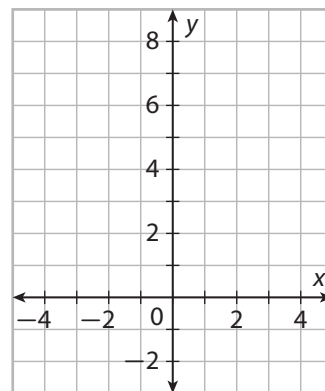
9.  $f(x) = \frac{(x + 1)(x - 1)}{x + 2}$



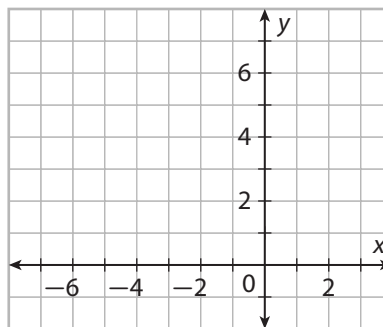
10.  $f(x) = \frac{-3x(x - 2)}{(x - 2)(x + 2)}$



11.  $f(x) = \frac{x^2 + 2x - 4}{x - 1}$



12.  $f(x) = \frac{2x^2 - 4x}{x^2 + 4x + 4}$



Write a rational function to model the situation, or use the given rational function. State a reasonable domain and range for the function using set notation. Then use a graphing calculator to graph the function and answer the question.

- 13.** A basketball team has won 16 games out of 23 games played, for a winning percentage (expressed as a decimal) of  $\frac{16}{23} \approx 0.696$ . How many consecutive games must the team win to raise its winning percentage to 0.750?



- 14.** So far this season, a baseball player has had 84 hits in 294 times at bat, for a batting average of  $\frac{84}{294} \approx 0.286$ . How many consecutive hits must the player get to raise his batting average to 0.300?

- 15.** A kayaker traveled 5 miles upstream and then 8 miles downstream on a river. The average speed of the current was 3 miles per hour. If the kayaker was paddling for 5 hours, what was the kayaker's average paddling speed? To answer the question, use the rational function  $t(s) = \frac{5}{s-3} + \frac{8}{s+3} = \frac{13s-9}{(s-3)(s+3)}$  where  $s$  is the kayaker's average paddling speed (in miles per hour) and  $t$  is the time (in hours) spent kayaking 5 miles against the current at a rate of  $s - 3$  miles per hour and 8 miles with the current at a rate of  $s + 3$  miles per hour.



- 16.** In aviation, *air speed* refers to a plane's speed in still air. A small plane maintains a certain air speed when flying to and from a city that is 400 miles away. On the trip out, the plane flies against a wind, which has an average speed of 40 miles per hour. On the return trip, the plane flies with the wind. If the total flight time for the round trip is 3.5 hours, what is the plane's average air speed? To answer this question, use the rational function  $t(s) = \frac{400}{s-40} + \frac{400}{s+40} = \frac{800s}{(s-40)(s+40)}$  air speed (in miles per hour) and  $t$  is the total flight time (in hours) for the round trip.



- 17. Multiple Response** Select the statements that apply to the rational function

$$f(x) = \frac{x - 2}{x^2 - x - 6}.$$

- A. The function's domain is  $\{x \mid x \neq -2 \text{ and } x \neq 3\}$ .
- B. The function's domain is  $\{x \mid x \neq -2 \text{ and } x \neq -3\}$ .
- C. The function's range is  $\{y \mid y \neq 0\}$ .
- D. The function's range is  $\{y \mid -\infty < y < +\infty\}$ .
- E. The function's graph has vertical asymptotes at  $x = -2$  and  $x = 3$ .
- F. The function's graph has a vertical asymptote at  $x = -3$  and a "hole" at  $x = 2$ .
- G. The function's graph has a horizontal asymptote at  $y = 0$ .
- H. The function's graph has a horizontal asymptote at  $y = 1$ .

**H.O.T. Focus on Higher Order Thinking**

- 18. Draw Conclusions** For what value(s) of  $a$  does the graph of  $f(x) = \frac{x + a}{x^2 + 4x + 3}$  have a "hole"? Explain. Then, for each value of  $a$ , state the domain and the range of  $f(x)$  using interval notation.

- 19. Critique Reasoning** A student claims that the functions  $f(x) = \frac{4x^2 - 1}{4x + 2}$  and  $g(x) = \frac{4x + 2}{4x^2 - 1}$  have different domains but identical ranges. Which part of the student's claim is correct, and which is false? Explain.

# Lesson Performance Task

In professional baseball, the smallest allowable volume of a baseball is 92.06% of the largest allowable volume, and the range of allowable radii is 0.04 inch.

- a. Let  $r$  be the largest allowable radius (in inches) of a baseball. Write expressions, both in terms of  $r$ , for the largest allowable volume of the baseball and the smallest allowable volume of the baseball. (Use the formula for the volume of a sphere,  $V = \frac{4}{3} \pi r^3$ .)
- b. Write and simplify a function that gives the ratio  $R$  of the smallest allowable volume of a baseball to the largest allowable volume.
- c. Use a graphing calculator to graph the function from part b, and use the graph to find the smallest allowable radius and the largest allowable radius of a baseball. Round your answers to the nearest hundredth.