Name ____________________ Class ____________ Date ________

8.1 Graphing Simple Rational Functions

Resource Locker

Essential Question: How are the graphs of $f(x) = a\left(\frac{1}{x-h}\right) + k$ and $f(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ related to the graph of $f(x) = \frac{1}{x}$?

Explore 1 Graphing and Analyzing $f(x) = \frac{1}{x}$

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, where $q(x) \neq 0$. The most basic rational function with a variable expression in the denominator is $f(x) = \frac{1}{x}$.

(A) State the domain of $f(x) = \frac{1}{x}$.

The function accepts all real numbers except ___, because division by ___ is undefined. So, the function's domain is as follows:

- As an inequality: $x < \square$ or $x > \square$
- In set notation: $\{x \mid x \neq \square\}$
- In interval notation (where the symbol ∪ means *union*):

$(-\infty, \square) \cup (\square, +\infty)$

(B) Determine the end behavior of $f(x) = \frac{1}{x}$.

First, complete the tables.

x Increases without Bound	
x	$f(x) = \frac{1}{x}$
100	
1000	
10,000	

x Decreases without Bound	
x	$f(x) = \frac{1}{x}$
−100	
−1000	
−10,000	

Next, summarize the results.

- As $x \to +\infty$, $f(x) \to \square$.
- As $x \to -\infty$, $f(x) \to \square$.

Ⓒ Be more precise about the end behavior of $f(x) = \frac{1}{x}$, and determine what this means for the graph of the function.

You can be more precise about the end behavior by using the notation $f(x) \to 0^+$, which means that the value of $f(x)$ approaches 0 from the positive direction (that is, the value of $f(x)$ is positive as it approaches 0), and the notation $f(x) \to 0^-$, which means that the value of $f(x)$ approaches 0 from the negative direction. So, the end behavior of the function is more precisely summarized as follows:

- As $x \to +\infty$, $f(x) \to$ ☐.
- As $x \to -\infty$, $f(x) \to$ ☐.

The end behavior indicates that the graph of $f(x)$ approaches, but does not cross, the [x-axis/y-axis], so that axis is an asymptote for the graph.

Ⓓ Examine the behavior of $f(x) = \frac{1}{x}$ near $x = 0$, and determine what this means for the graph of the function.

First, complete the tables.

***x* Approaches 0 from the Positive Direction**	
x	$f(x) = \frac{1}{x}$
0.01	
0.001	
0.0001	

***x* Approaches 0 from the Negative Direction**	
x	$f(x) = \frac{1}{x}$
−0.01	
−0.001	
−0.0001	

Next, summarize the results.

- As $x \to 0^+$, $f(x) \to$ ☐.
- As $x \to 0^-$, $f(x) \to$ ☐.

The behavior of $f(x) = \frac{1}{x}$ near $x = 0$ indicates that the graph of $f(x)$ approaches, but does not cross, the [x-axis/y-axis], so that axis is also an asymptote for the graph.

Ⓔ Graph $f(x) = \frac{1}{x}$.

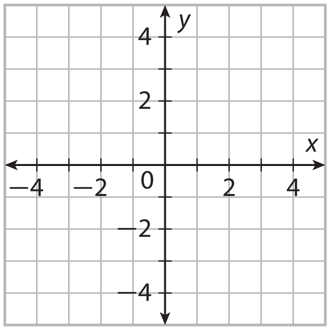
First, determine the sign of $f(x)$ on the two parts of its domain.

- When x is a negative number, $f(x)$ is a [positive/negative] number.
- When x is a positive number, $f(x)$ is a [positive/negative] number.

Next, complete the tables.

Negative Values of *x*	
x	$f(x) = \frac{1}{x}$
−2	
−1	
−0.5	

Positive Values of *x*	
x	$f(x) = \frac{1}{x}$
0.5	
1	
2	



Finally, use the information from this step and all previous steps to draw the graph. Draw asymptotes as dashed lines.

(F) State the range of $f(x) = \frac{1}{x}$.

The function takes on all real numbers except ________, so the function's range is as follows:

- As an inequality: $y < \square$ or $y > \square$
- In set notation: $\{y | y \neq \square\}$
- In interval notation (where the symbol $\cup$ means *union*): $(-\infty, \square) \cup (\square, +\infty)$

(G) Identify the intervals where the function is increasing and where it is decreasing.

(H) Determine whether $f(x) = \frac{1}{x}$ is an even function, an odd function, or neither.

Reflect

1. How does the graph of $f(x) = \frac{1}{x}$ show that the function has no zeros?

2. **Discussion** A graph is said to be *symmetric about the origin* (and the origin is called the graph's *point of symmetry*) if for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. Is the graph of $f(x) = \frac{1}{x}$ symmetric about the origin? Explain.

3. Give any line(s) of symmetry for the graph of $f(x) = \frac{1}{x}$.

Explain 1 Graphing Simple Rational Functions

When graphing transformations of $f(x) = \frac{1}{x}$, it helps to consider the effect of the transformations on the following features of the graph of $f(x)$: the vertical asymptote, $x = 0$; the horizontal asymptote, $y = 0$; and two reference points, $(-1, -1)$ and $(1, 1)$. The table lists these features of the graph of $f(x)$ and the corresponding features of the graph of $g(x) = a\left(\frac{1}{\frac{1}{b}(x-h)}\right) + k$. Note that the asymptotes are affected only by the parameters h and k, while the reference points are affected by all four parameters.

Feature	$f(x) = \frac{1}{x}$	$g(x) = a\left(\frac{1}{\frac{1}{b}(x-h)}\right) + k$
Vertical asymptote	$x = 0$	$x = h$
Horizontal asymptote	$y = 0$	$y = k$
Reference point	$(-1, -1)$	$(-b + h, -a + k)$
Reference point	$(1, 1)$	$(b + h, a + k)$

Example 1 **Identify the transformations of the graph of $f(x) = \frac{1}{x}$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.**

(A) $g(x) = 3\left(\frac{1}{x-1}\right) + 2$

The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:

- a vertical stretch by a factor of 3
- a translation of 1 unit to the right and 2 units up

Note that the translation of 1 unit to the right affects only the x-coordinates, while the vertical stretch by a factor of 3 and the translation of 2 units up affect only the y-coordinates.

Feature	$f(x) = \frac{1}{x}$	$g(x) = 3\left(\frac{1}{x-1}\right) + 2$
Vertical asymptote	$x = 0$	$x = 1$
Horizontal asymptote	$y = 0$	$y = 2$
Reference point	$(-1, -1)$	$(-1 + 1, 3(-1) + 2) = (0, -1)$
Reference point	$(1, 1)$	$(1 + 1, 3(1) + 2) = (2, 5)$

Domain of $g(x)$:

Inequality: $x < 1$ or $x > 1$

Set notation: $\{x \mid x \neq 1\}$

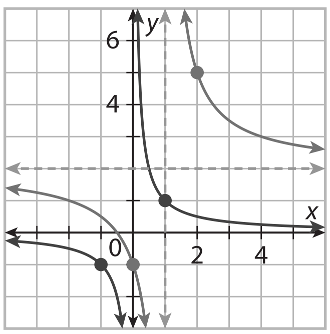
Interval notation: $(-\infty, 1) \cup (1, +\infty)$

Range of $g(x)$:

Inequality: $y < 2$ or $y > 2$

Set notation: $\{y \mid y \neq 2\}$

Interval notation: $(-\infty, 2) \cup (2, +\infty)$



(B) $g(x) = \dfrac{1}{2(x+3)} - 1$

The transformations of the graph of $f(x)$ that produce the graph of $g(x)$ are:

- a horizontal compression by a factor of $\frac{1}{2}$
- a translation of 3 units to the left and 1 unit down

Note that the horizontal compression by a factor of $\frac{1}{2}$ and the translation of 3 units to the left affect only the x-coordinates of points on the graph of $f(x)$, while the translation of 1 unit down affects only the y-coordinates.

Feature	$f(x) = \frac{1}{x}$	$g(x) = \frac{1}{2(x+3)} - 1$
Vertical asymptote	$x = 0$	$x = \square$
Horizontal asymptote	$y = 0$	$y = \square$
Reference point	$(-1, -1)$	$\left(\frac{1}{2}(\square) - 3, \square - 1\right) = \left(\square, \square\right)$
Reference point	$(1, 1)$	$\left(\frac{1}{2}(\square) - 3, \square - 1\right) = \left(\square, \square\right)$

Domain of $g(x)$:

Inequality: $x < \square$ or $x > \square$

Set notation: $\{x \mid x \neq \square\}$

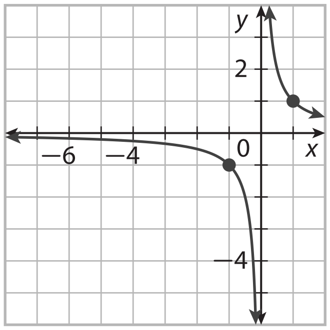
Interval notation: $(-\infty, \square) \cup (\square, +\infty)$

Range of $g(x)$:

Inequality: $y < \square$ or $y > \square$

Set notation: $\{y \mid y \neq \square\}$

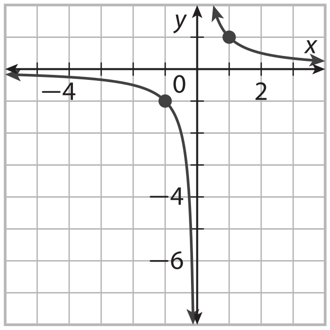
Interval notation: $(-\infty, \square) \cup (\square, +\infty)$



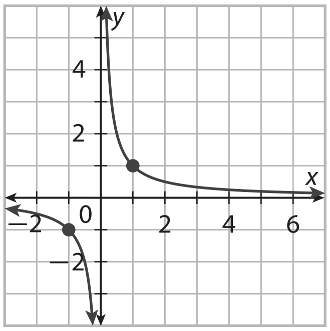
Your Turn

Identify the transformations of the graph of $f(x) = \frac{1}{x}$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.

4. $g(x) = -0.5\left(\frac{1}{x+1}\right) - 3$



5. $g(x) = \frac{1}{-0.5(x-2)} + 1$



Explain 2 Rewriting Simple Rational Functions in Order to Graph Them

When given a rational function of the form $g(x) = \frac{mx+n}{px+q}$, where $m \neq 0$ and $p \neq 0$, you can carry out the division of the numerator by the denominator to write the function in the form $g(x) = a\left(\frac{1}{x-h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ in order to graph it.

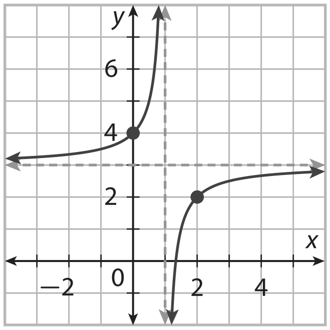
Example 2 **Rewrite the function in the form $g(x) = a\left(\frac{1}{x-h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$ and graph it. Also state the domain and range using inequalities, set notation, and interval notation.**

(A) $g(x) = \dfrac{3x-4}{x-1}$

Use long division.

$$\begin{array}{r} 3 \\ x-1\overline{)\,3x-4} \\ \underline{3x-3} \\ -1 \end{array}$$

So, the quotient is 3, and the remainder is −1. Using the fact that *dividend* = *quotient* + $\frac{\textit{remainder}}{\textit{divisor}}$, you have $g(x) = 3 + \frac{-1}{x-1}$, or $g(x) = -\frac{1}{x-1} + 3$.



The graph of $g(x)$ has vertical asymptote $x = 1$, horizontal asymptote $y = 3$, and reference points $\left(-1+1, -(-1)+3\right) = (0, 4)$ and $(1+1, -(1)+3) = (2, 2)$.

Domain of $g(x)$:

Inequality: $x < 1$ or $x > 1$

Set notation: $\{x \mid x \neq 1\}$

Interval notation: $(-\infty, 1) \cup (1, +\infty)$

Range of $g(x)$:

Inequality: $y < 3$ or $y > 3$

Set notation: $\{y \mid y \neq 3\}$

Interval notation: $(-\infty, 3) \cup (3, +\infty)$

(B) $g(x) = \dfrac{4x-7}{-2x+4}$

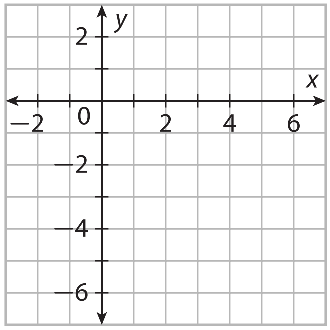
Use long division.

$$\begin{array}{r} -2 \\ -2x+4\overline{)\,4x-7} \\ \underline{4x-8} \\ \square \end{array}$$

So, the quotient is −2, and the remainder is ____. Using the fact that *dividend* = *quotient* + $\frac{\textit{remainder}}{\textit{divisor}}$, you have

$$g(x) = -2 + \frac{\square}{-2x+4}, \text{ or } g(x) = \frac{\square}{-2\left(x - \square\right)} - 2.$$

The graph of $g(x)$ has vertical asymptote $x = \square$, horizontal asymptote $y = -2$, and reference points $\left(-\frac{1}{2}(-1) + \square, -1 - 2\right) = \left(\square, -3\right)$ and $\left(-\frac{1}{2}(1) + \square, 1 - 2\right) = \left(\square, -1\right)$.



Domain of $g(x)$:

Inequality: $x < \square$ or $x > \square$

Set notation: $\left\{x \mid x \neq \square\right\}$

Interval notation: $\left(-\infty, \square\right) \cup \left(\square, +\infty\right)$

Range of $g(x)$:

Inequality: $y < \square$ or $y > \square$

Set notation: $\left\{y \mid y \neq \square\right\}$

Interval notation: $\left(-\infty, \square\right) \cup \left(\square, +\infty\right)$

Reflect

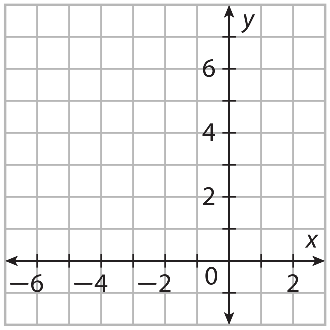
6. In Part A, the graph of $g(x)$ is the result of what transformations of the graph of $f(x) = \frac{1}{x}$?

7. In Part B, the graph of $g(x)$ is the result of what transformations of the graph of $f(x) = \frac{1}{x}$?

Your Turn

Rewrite the function in the form $g(x) = a\left(\frac{1}{x - h}\right) + k$ or $g(x) = \frac{1}{\frac{1}{b}(x - h)} + k$ and graph it. Also state the domain and range using inequalities, set notation, and interval notation.

8. $g(x) = \dfrac{3x + 8}{x + 2}$



Explain 3 Writing Simple Rational Functions

When given the graph of a simple rational function, you can write its equation using one of the general forms $g(x) = a\left(\frac{1}{x - h}\right) + k$ and $g(x) = \frac{1}{\frac{1}{b}(x - h)} + k$ after identifying the values of the parameters using information obtained from the graph.

Example 3

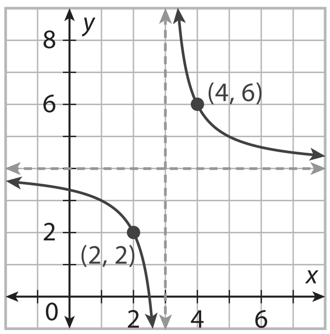
(A) **Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x - h}\right) + k$.**

Since the graph's vertical asymptote is $x = 3$, the value of the parameter h is 3. Since the graph's horizontal asymptote is $y = 4$, the value of the parameter k is 4.

Substitute these values into the general form of the function.

$$g(x) = a\left(\frac{1}{x - 3}\right) + 4$$

Now use one of the points, such as $(4, 6)$, to find the value of the parameter a.

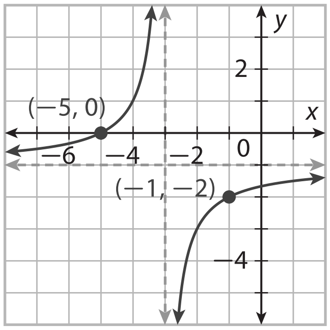


$$g(x) = a\left(\frac{1}{x - 3}\right) + 4$$
$$6 = a\left(\frac{1}{4 - 3}\right) + 4$$
$$6 = a + 4$$
$$2 = a$$

So, $g(x) = 2\left(\frac{1}{x - 3}\right) + 4$.

Ⓑ Write the function whose graph is shown. Use the form $g(x) = \dfrac{1}{\frac{1}{b}(x-h)} + k$.

Since the graph's vertical asymptote is $x = -3$, the value of the parameter h is -3. Since the graph's horizontal asymptote is $y = \square$, the value of the parameter k is ____.



Substitute these values into the general form of the function.

$$g(x) = \frac{1}{\frac{1}{b}(x+3)} + \square$$

Now use one of the points, such as $(-5, 0)$, to find the value of the parameter a.

$$g(x) = \frac{1}{\frac{1}{b}(x+3)} + \square$$

$$0 = \frac{1}{\frac{1}{b}(-5+3)} + \square$$

$$\square = \frac{1}{\frac{1}{b}(-2)}$$

$$\frac{1}{b}(-2) \cdot \square = 1$$

$$\frac{1}{b} = \square$$

$$b = \square$$

So, $g(x) = \dfrac{1}{\frac{1}{\square}(x+3)} + \square$, or $g(x) = \dfrac{1}{-0.5(x+3)} - 1$.

Reflect

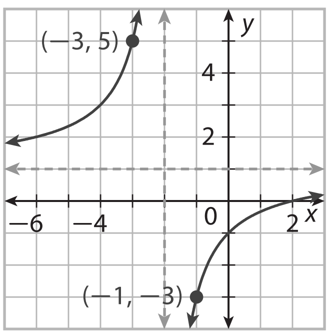
9. Discussion In Parts A and B, the coordinates of a second point on the graph of $g(x)$ are given. In what way can those coordinates be useful?

____

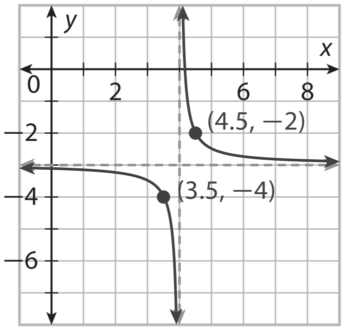
____

Your Turn

10. Write the function whose graph is shown. Use the form $g(x) = a\left(\dfrac{1}{x-h}\right) + k$.



11. Write the function whose graph is shown. Use the form $g(x) = \dfrac{1}{\frac{1}{b}(x - h)} + k$.



Explain 4 Modeling with Simple Rational Functions

In a real-world situation where there is a shared cost and a per-person or per-item cost, you can model the situation using a rational function that has the general form $f(x) = \frac{a}{x - h} + k$ where $f(x)$ is the total cost for each person or item.

Example 4

(A) **Mary and some of her friends are thinking about renting a car while staying at a beach resort for a vacation. The cost per person for staying at the beach resort is \$300, and the cost of the car rental is \$220. If the friends agree to share the cost of the car rental, what is the minimum number of people who must go on the trip so that the total cost for each person is no more than \$350?**

Analyze Information

Identify the important information.

- The cost per person for the resort is ______.
- The cost of the car rental is ______.
- The most that each person will spend is ______

Formulate a Plan

Create a rational function that gives the total cost for each person. Graph the function, and use the graph to answer the question.

Solve

Let p be the number of people who agree to go on the trip. Let $C(p)$ be the cost (in dollars) for each person.

$$C(p) = \frac{\boxed{}}{p} + \boxed{}$$

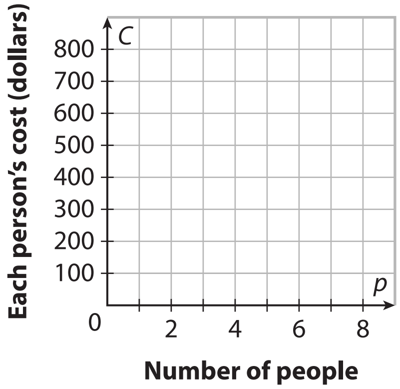
Graph the function, recognizing that the graph involves two transformations of the graph of the parent rational function:

- a vertical stretch by a factor of ______
- a vertical translation of ______ units up

Also draw the line $C(p) = 350$.

The graphs intersect between $p = \boxed{}$ and $p = \boxed{}$, so the minimum number of people who must go on the trip in order for the total cost for each person to be no more than \$350

is ____.

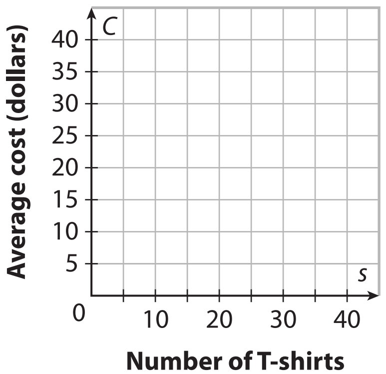


Justify and Evaluate

Check the solution by evaluating the function $C(p)$. Since $C(4) = \boxed{} > 350$ and $C(5) = \boxed{} < 350$, the minimum number of people who must go on the trip is ______.

Your Turn

12. Justin has purchased a basic silk screening kit for applying designs to fabric. The kit costs \$200. He plans to buy T-shirts for \$10 each, apply a design that he creates to them, and then sell them. Model this situation with a rational function that gives the average cost of a silk-screened T-shirt when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of T-shirts that brings the average cost below \$17.50.



Elaborate

13. Compare and contrast the attributes of $f(x) = \frac{1}{x}$ and the attributes of $g(x) = -\frac{1}{x}$.

14. State the domain and range of $f(x) = a\left(\frac{1}{x - h}\right) + k$ using inequalities, set notation, and interval notation.

15. Given that the model $C(p) = \frac{100}{p} + 50$ represents the total cost C (in dollars) for each person in a group of p people when there is a shared expense and an individual expense, describe what the expressions $\frac{100}{p}$ and 50 represent.

16. **Essential Question Check-In** Describe the transformations you must perform on the graph of $f(x) = \frac{1}{x}$ to obtain the graph of $f(x) = a\left(\frac{1}{x - h}\right) + k$.

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Describe how the graph of $g(x)$ is related to the graph of $f(x) = \frac{1}{x}$.

1. $g(x) = \frac{1}{x} + 4$

2. $g(x) = 5\left(\frac{1}{x}\right)$

3. $g(x) = \dfrac{1}{x + 3}$

4. $g(x) = \dfrac{1}{0.1x}$

5. $g(x) = \frac{1}{x} - 7$

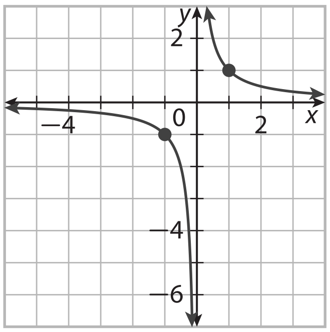
6. $g(x) = \dfrac{1}{x - 8}$

7. $g(x) = -0.1\left(\frac{1}{x}\right)$

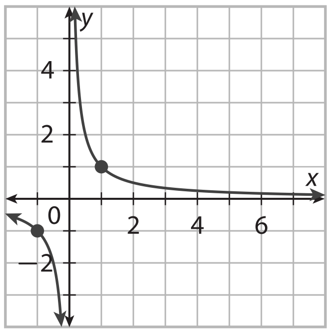
8. $g(x) = \dfrac{1}{-3x}$

Identify the transformations of the graph of $f(x)$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptotes $x = 0$ and $y = 0$ and to the reference points $(-1, -1)$ and $(1, 1)$. Also state the domain and range of $g(x)$ using inequalities, set notation, and interval notation.

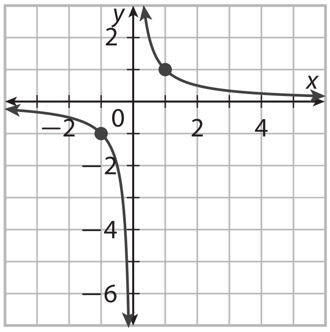
9. $g(x) = 3\left(\dfrac{1}{x + 1}\right) - 2$



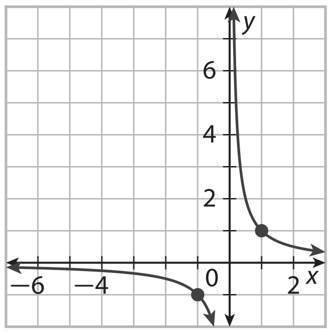
10. $g(x) = \dfrac{1}{-0.5(x-3)} + 1$



11. $g(x) = -0.5\left(\dfrac{1}{x-1}\right) - 2$

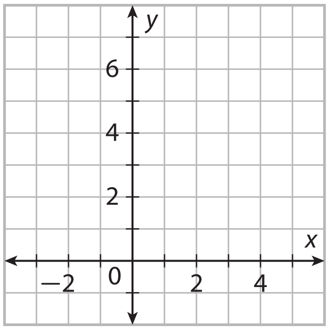


12. $g(x) = \dfrac{1}{2(x+2)} + 3$

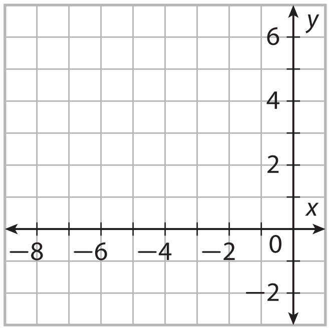


Rewrite the function in the form $g(x) = a\dfrac{1}{(x-h)} + k$ or $g(x) = \dfrac{1}{\frac{1}{b}(x-h)} + k$ and graph it.
Also state the domain and range using inequalities, set notation, and interval notation.

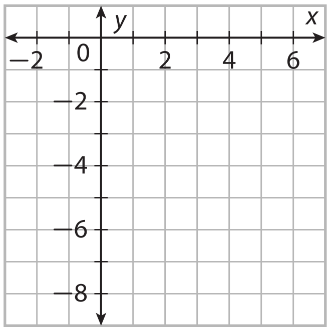
13. $g(x) = \dfrac{3x-5}{x-1}$



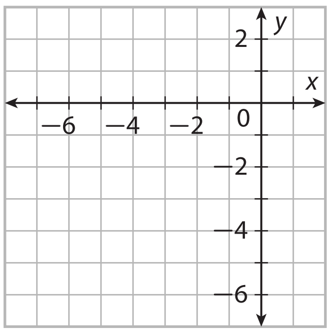
14. $g(x) = \dfrac{x+5}{0.5x+2}$



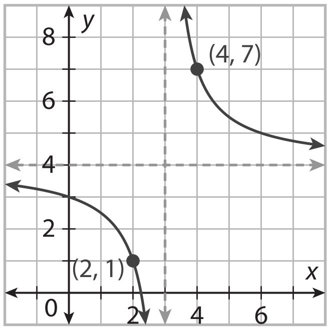
15. $g(x) = \dfrac{-4x+11}{x-2}$



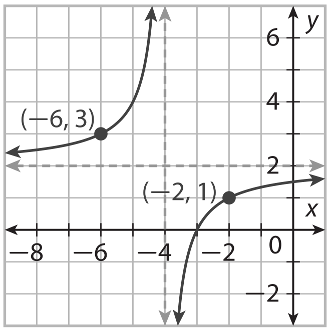
16. $g(x) = \dfrac{4x + 13}{-2x - 6}$



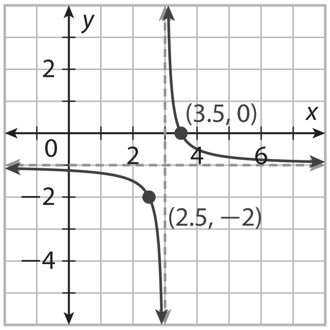
17. Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x - h}\right) + k$.



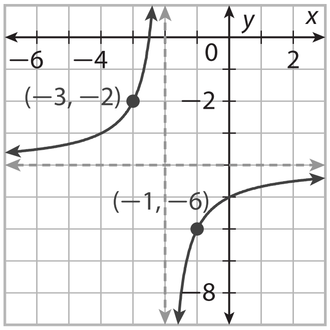
18. Write the function whose graph is shown. Use the form $g(x) = \dfrac{1}{\frac{1}{b}(x - h)} + k$



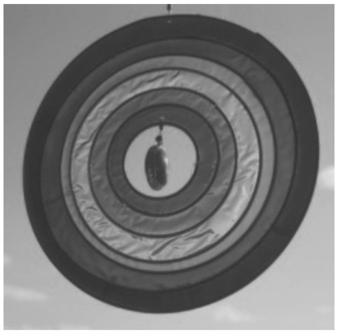
19. Write the function whose graph is shown. Use the form $g(x) = \frac{1}{\frac{1}{b}(x-h)} + k$.

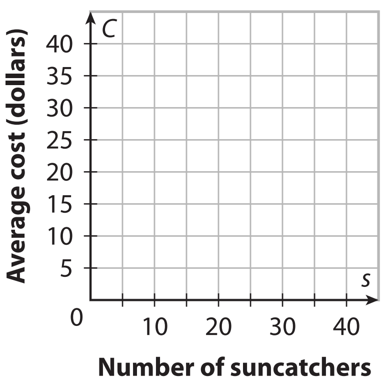


20. Write the function whose graph is shown. Use the form $g(x) = a\left(\frac{1}{x-h}\right) + k$

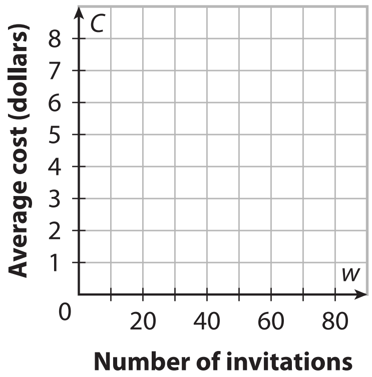


21. Maria has purchased a basic stained glass kit for \$100. She plans to make stained glass suncatchers and sell them. She estimates that the materials for making each suncatcher will cost \$15. Model this situation with a rational function that gives the average cost of a stained glass suncatcher when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of suncatchers that brings the average cost below \$22.50.





22. Amy has purchased a basic letterpress kit for $140. She plans to make wedding invitations. She estimates that the cost of the paper and envelope for each invitation is $2. Model this situation with a rational function that gives the average cost of a wedding invitation when the cost of the kit is included in the calculation. Use the graph of the function to determine the minimum number of invitations that brings the average cost below $5.



23. Multiple Response Select the transformations of the graph of the parent rational function that result in the graph of $g(x) = \frac{1}{2(x-3)} + 1$

A. Horizontal stretch by a factor of 2

B. Horizontal compression by a factor of $\frac{1}{2}$

C. Vertical stretch by a factor of 2

D. Vertical compression by a factor of $\frac{1}{2}$

E. Translation 1 unit up

F. Translation 1 unit down

G. Translation 3 units right

H. Translation 3 units left

H.O.T. Focus on Higher Order Thinking

24. Justify Reasoning Explain why, for positive numbers a and b, a vertical stretch or compression of the graph of $f(x) = \frac{1}{x}$ by a factor of a and, separately, a horizontal stretch or compression of the graph of $f(x)$ by a factor of b result in the same graph when a and b are equal.

25. Communicate Mathematical Ideas Determine the domain and range of the rational function $g(x) = \frac{mx+n}{px+q}$ where $p \neq 0$. Give your answer in set notation, and explain your reasoning.

Lesson Performance Task

Graham wants to take snowboarding lessons at a nearby ski resort that charges $40 per week for a class that meets once during the week for 1 hour and gives him a lift ticket valid for 4 hours. The resort also charges a one-time equipment-rental fee of $99 for uninterrupted enrollment in classes. The resort requires that learners pay for three weeks of classes at a time.

a. Write a model that gives Graham's average weekly enrollment cost (in dollars) as a function of the time (in weeks) that Graham takes classes.

b. How much would Graham's average weekly enrollment cost be if he took classes only for the minimum of three weeks?

c. For how many weeks would Graham need to take classes for his average weekly enrollment cost to be at most $60? Describe how you can use a graphing calculator to graph the function from part a in order to answer this question, and then state the answer.