Name ______________________ Class ____________ Date ________

19.2 Permutations and Probability

Essential Question: When are permutations useful in calculating probability?

Resource Locker

Explore Finding the Number of Permutations

A **permutation** is a selection of objects from a group in which order is important. For example, there are 6 permutations of the letters A, B, and C.

ABC	ACB	BAC	BCA	CAB	CBA

You can find the number of permutations with the **Fundamental Counting Principle**.

Fundamental Counting Principle

If there are n items and a_1 ways to choose the first item, a_2 ways to select the second item after the first item has been chosen, and so on, there are $a_1 \times a_2 \times \ldots \times a_n$ ways to choose n items.

There are 7 members in a club. Each year the club elects a president, a vice president, and a treasurer.

(A) What is the number of permutations of all 7 members of the club?

There are ____ different ways to make the first selection.

Once the first person has been chosen, there are ____ different ways to make the second selection.

Once the first two people have been chosen, there are ____ different ways to make the third selection.

Continuing this pattern, there are ______________________ permutations of all the members of the club.

(B) The club is holding elections for a president, a vice president, and a treasurer. How many different ways can these positions be filled?

There are ____ different ways the position of president can be filled.

Once the president has been chosen, there are ____ different ways the position of vice president can be filled. Once the president and vice president have been chosen, there are ____ different ways the position of treasurer can be filled.

So, there are ______________ different ways that the positions can be filled.

Ⓒ What is the number of permutations of the members of the club who were not elected as officers?

After the officers have been elected, there are ____ members remaining. So there are ____ different ways to make the first selection.

Once the first person has been chosen, there are ____ different ways to make the second selection.

Continuing this pattern, there are ____________________ permutations of the unelected members of the club.

Ⓓ Divide the number of permutations of all the members by the number of permutations of the unelected members.

There are ______ permutations of all the members of the club.

There are ____ permutations of the unelected members of the club.

The quotient of these two values is __________.

Reflect

1. How does the answer to Step D compare to the answer to Step B?

2. **Discussion** Explain the effect of dividing the total number of permutations by the number of permutations of items not selected.

Explain 1 Finding a Probability Using Permutations

The results of the Explore can be generalized to give a formula for permutations. To do so, it is helpful to use *factorials.* For a positive integer n, n **factorial,** written $n!$, is defined as follows.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

That is, $n!$ is the product of n and all the positive integers less than n. Note that 0! is defined to be 1.

In the Explore, the number of permutations of the 7 objects taken 3 at a time is

$$7 \times 6 \times 5 = \frac{7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$$

This can be generalized as follows.

Permutations

The number of permutations of n objects taken r at a time is given by ${}_nP_r = \frac{n!}{(n-r)!}$.

Example 1 **Use permutations to find the probabilities.**

(A) A research laboratory requires a four-digit security code to gain access to the facility. A security code can contain any of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, but no digit is repeated. What is the probability that a scientist is randomly assigned a code with the digits 1, 2, 3, and 4 in any order?

The sample space S consists is the number of permutations of 4 digits selected from 10 digits.

$$n(S) = {}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

Event A consists of permutations of a security code with the digits 1, 2, 3, and 4.

$$n(A) = {}_4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 24$$

The probability of getting a security code with the digits 1, 2, 3, and 4 is

$$P(A) = \frac{n(A)}{n(S)} = \frac{24}{5040} = \frac{1}{210}.$$

(B) A certain motorcycle license plate consists of 5 digits that are randomly selected. No digit is repeated. What is the probability of getting a license plate consisting of all even digits?

The sample space S consists of permutations of ________ selected from ________.

$$n(S) = {}_{\square}P_{\square} = \frac{\square}{\square} = \square$$

Event A consists of permutations of a license plate with ____________.

$$n(A) = {}_{\square}P_{\square}\ \frac{\square}{\square} = \square$$

The probability of getting a license plate with ____________ is

$$P(A) = \frac{n(A)}{n(S)} = \frac{\square}{\square} = \frac{\square}{\square}.$$

Your Turn

There are 8 finalists in the 100-meter dash at the Olympic Games. Suppose 3 of the finalists are from the United States, and that all finalists are equally likely to win.

3. What is the probability that the United States will win all 3 medals in this event?

4. What is the probability that the United States will win no medals in this event?

Explain 2 Finding the Number of Permutations with Repetition

Up to this point, the problems have focused on finding the permutations of distinct objects. If some of the objects are repeated, this will reduce the number of permutations that are distinguishable.

For example, here are the permutations of the letters A, B, and C.

ABC	ACB	BAC	BCA	CAB	CBA

Next, here are the permutations of the letters M, O, and M. Bold type is used to show the different positions of the repeated letter.

MOM	MO**M**	**M**MO	M**M**O	O**M**M	OM**M**

Shown without the bold type, here are the permutations of the letters M, O, and M.

MOM	MOM	MMO	MMO	OMM	OMM

Notice that since the letter M is repeated, there are only 3 distinguishable permutations of the letters. This can be generalized with a formula for permutations with repetition.

Permutations with Repetition

The number of different permutations of n objects where one object repeats a times, a second object repeats b times, and so on is

$$\frac{n!}{a! \times b! \times \ldots}$$

Example 2 **Find the number of permutations.**

(A) How many different permutations are there of the letters in the word ARKANSAS?

There are 8 letters in the word, and there are 3 A's and 2 S's, so the number of permutations of the letters in ARKANSAS is $\frac{8!}{3!2!} = 3360$.

(B) One of the zip codes for Anchorage, Alaska, is 99522. How many permutations are there of the numbers in this zip code?

There are ____ digits in the zip code, and there are ________, and ________ in the zip code, so the number of permutations of the zip code is

$$\frac{\boxed{}}{\boxed{}} = \boxed{}.$$

Your Turn

5. How many different permutations can be formed using all the letters in MISSISSIPPI?

6. One of the standard telephone numbers for directory assistance is 555–1212. How many different permutations of this telephone number are possible?

Explain 3 Finding a Probability Using Permutations with Repetition

Permutations with repetition can be used to find probablilities.

Example 3 **The school jazz band has 4 boys and 4 girls, and they are randomly lined up for a yearbook photo.**

(A) Find the probability of getting an alternating boy-girl arrangement.

The sample space S consists of permutations of 8 objects, with 4 boys and 4 girls.

$n(S)\ \frac{8!}{4!4!} = 70$

Event A consists of permutations that alternate boy-girl or girl-boy. The possible permutations are BGBGBGBG and GBGBGBGB.

$n(A) = 2$

The probability of getting an alternating boy-girl arrangement is $P(A) = \frac{n(A)}{n(S)} = \frac{2}{70} = \frac{1}{35}$.

(B) Find the probability of getting all of the boys grouped together.

The sample space S consists of permutations of ________, with ________________.

$n(S) = \frac{\square}{\square} = \square$

Event A consists of permutations with ________________. The possible permutations

are BBBBGGGG, GBBBBGGG, ________________________________.

$n(A) = \square$

The probability of getting all the boys grouped together is $P(A) = \frac{n(A)}{n(S)} = \frac{\square}{\square} = \frac{\square}{\square}$.

Your Turn

7. There are 2 mystery books, 2 romance books, and 2 poetry books to be randomly placed on a shelf. What is the probability that the mystery books are next to each other, the romance books are next to each other, and the poetry books are next to each other?

8. What is the probability that a random arrangement of the letters in the word APPLE will have the two P's next to each other?

Elaborate

9. If ${}_nP_a = {}_nP_b$, what is the relationship between a and b? Explain your answer.

10. It was observed that there are 6 permutations of the letters A, B, and C. They are ABC, ACB, BAC, BCA, CAB, and CBA. If the conditions are changed so that the order of selection does not matter, what happens to these 6 different groups?

11. **Essential Question Check-In** How do you determine whether choosing a group of objects involves permutations?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. An MP3 player has a playlist with 12 songs. You select the shuffle option, which plays each song in a random order without repetition, for the playlist. In how many different orders can the songs be played?

2. There are 10 runners in a race. Medals are awarded for 1st, 2nd, and 3rd place. In how many different ways can the medals be awarded?

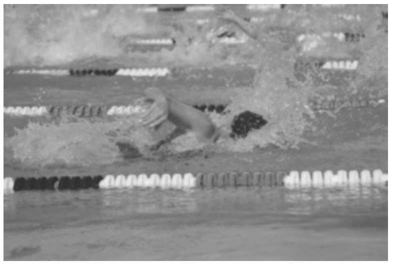
3. There are 9 players on a baseball team. In how many different ways can the coach choose players for first base, second base, third base, and shortstop?

4. A bag contains 9 tiles, each with a different number from 1 to 9. You choose a tile without looking, put it aside, choose a second tile without looking, put it aside, then choose a third tile without looking. What is the probability that you choose tiles with the numbers 1, 2, and 3 in that order?

5. There are 11 students on a committee. To decide which 3 of these students will attend a conference, 3 names are chosen at random by pulling names one at a time from a hat. What is the probability that Sarah, Jamal, and Mai are chosen in any order?

6. A clerk has 4 different letters that need to go in 4 different envelopes. The clerk places one letter in each envelope at random. What is the probability that all 4 letters are placed in the correct envelopes?

7. A swim coach randomly selects 3 swimmers from a team of 8 to swim in a heat. What is the probability that she will choose the three strongest swimmers?



8. How many different sequences of letters can be formed using all the letters in ENVELOPE?

9. Yolanda has 3 each of red, blue, and green marbles. How many possible ways can the 9 marbles be arranged in a row?

10. Jane has 16 cards. Ten of the cards look exactly the same and have the number 1 on them. The other 6 cards look exactly the same and have the number 2 on them. Jane is going to make a row containing all 16 cards. How many different ways can she order the row?

11. Ramon has 10 cards, each with one number on it. The numbers are 1, 2, 3, 4, 4, 6, 6, 6, 6, 6. Ramon is going to make a row containing all 10 cards. How many different ways can he order the row?

12. A grocer has 5 apples and 5 oranges for a window display. The grocer makes a row of the 10 pieces of fruit by choosing one piece of fruit at random, making it the first piece in the row, choosing a second piece of fruit at random, making it the second piece in the row, and so on. What is the probability that the grocer arranges the fruits in alternating order? (Assume that the apples are not distinguishable and that the oranges are not distinguishable.)

13. The letters G, E, O, M, E, T, R, Y are on 8 tiles in a bag, one letter on each tile. If you select tiles randomly from the bag and place them in a row from left to right, what is the probability the tiles will spell out GEOMETRY?

14. There are 11 boys and 10 girls in a classroom. A teacher chooses a student at random and puts that student at the head of a line, chooses a second student at random and makes that student second in the line, and so on, until all 21 students are in the line. What is the probability that the teacher puts them in a line alternating boys and girls? where no two of the same gender stand together?

15. There are 4 female and 4 male kittens are sleeping together in a row. Assuming that the arrangement is a random arrangement, what is the probability that all the female kittens are together, and all the male kittens are together?

16. If a ski club with 12 members votes to choose 3 group leaders, what is the probability that Marsha, Kevin, and Nicola will be chosen in any order for President, Treasurer, and Secretary?

17. There are 7 books numbered 1–7 on the summer reading list. Peter randomly chooses 2 books. What is the probability that Peter chooses books numbered 1 and 2, in either order?

18. On an exam, students are asked to list 5 historical events in the order in which they occurred. A student randomly orders the events. What is the probability that the student chooses the correct order?

19. A fan makes 6 posters to hold up at a basketball game. Each poster has a letter of the word TIGERS. Six friends sit next to each other in a row. The posters are distributed at random. What is the probability that TIGERS is spelled correctly when the friends hold up the posters?

20. The 10 letter tiles S, A, C, D, E, E, M, I, I, and O are in a bag. What is the probability that the letters S-A-M-E will be drawn from the bag at random, in that order?

21. If three cards are drawn at random from a standard deck of 52 cards, what is the probability that they will all be 7s? (There are four 7s in a standard deck of 52 cards.)

22. A shop classroom has ten desks in a row. If there are 6 students in shop class and they choose their desks at random, what is the probability they will sit in the first six desks?

23. Match each event with its probability. All orders are chosen randomly.

A. There are 15 floats that will be in a town parade. Event *A*: The mascot float is chosen to be first and the football team float is chosen to be second.

B. Beth is one of 10 students performing in a school talent show. Event *B*: Beth is chosen to be the fifth performer and her best friend is chosen to be fourth.

C. Sylvester is in a music competition with 14 other musicians. Event *C*: Sylvester is chosen to be last, and his two best friends are chosen to be first and second.

_____ $\frac{1}{1092}$

_____ $\frac{1}{210}$

_____ $\frac{1}{90}$

H.O.T. Focus on Higher Order Thinking

24. Explain the Error Describe and correct the error in evaluating the expression.

$${}_5P_3 = \frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 20$$

25. Make a Conjecture If you are going to draw four cards from a deck of cards, does drawing four aces from the deck have the same probability as drawing four 3s? Explain.

26. Communicate Mathematical Ideas Nolan has Algebra, Biology, and World History homework. Assume that he chooses the order that he does his homework at random. Explain how to find the probability of his doing his Algebra homework first.

27. **Explain the Error** A student solved the problem shown. The student's work is also shown. Explain the error and provide the correct answer.

A bag contains 6 tiles with the letters A, B, C, D, E, and F, one letter on each tile. You choose 4 tiles one at a time without looking and line them up from left to right as you choose them. What is the probability that your tiles spell BEAD?

Let S be the sample space and let A be the event that the tiles spell BEAD.

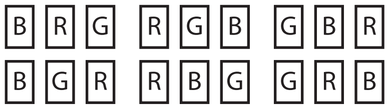
$$n(S) = {}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$$

$$n(A) = {}_4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 24$$

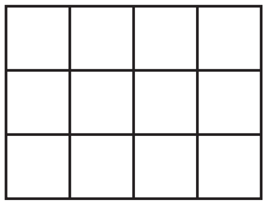
$$P(A) = \frac{n(A)}{n(S)} = \frac{24}{360} = \frac{1}{5}$$

Lesson Performance Task

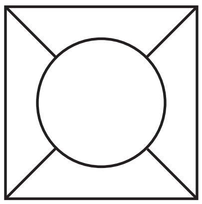
How many different ways can a blue card, a red card, and a green card be arranged? The diagram shows that the answer is six.



1. Now solve this problem: What is the least number of colors needed to color the pattern shown here, so that no two squares with a common boundary have the same color? Draw a sketch to show your answer.



2. Now try this one. Again, find the least number of colors needed to color the pattern so that no two regions with a common boundary have the same color. Draw a sketch to show your answer.



3. In 1974, Kenneth Appel and Wolfgang Haken solved a problem that had confounded mathematicians for more than a century. They proved that no matter how complex a map is, it can be colored in a maximum of four colors, so that no two regions with a common boundary have the same color. Sketch the figure shown here. Can you color it in four colors? Can you color it in three colors?