19.1 Probability and Set Theory

Essential Question: How are sets and their relationships used to calculate probabilities?

⊘ Explore · Working with Sets

A **set** is a collection of distinct objects. Each object in a set is called an **element** of the set. A set is often denoted by writing the elements in braces.

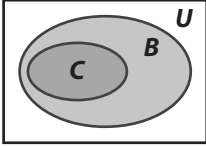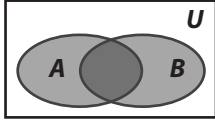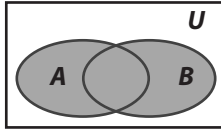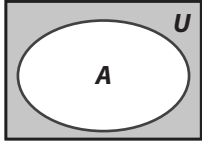The set with no elements is the **empty set**, denoted by ∅ or **{ }**.

The set of all elements under consideration is the **universal set**, denoted by *U*.

Identifying the number of elements in a set is important for calculating probabilities.

(A) Use set notation to identify each set described in the table and identify the number of elements in each set.

Set	Set Notation	Number of Elements in the Set
Set *A* is the set of prime numbers less than 10.	$A = \left\{ 2, 3, \boxed{}, 7 \right\}$	$n(A) = 4$
Set *B* is the set of even natural numbers less than 10.	$B = \left\{ \boxed{}, \boxed{}, \boxed{}, \boxed{} \right\}$	$n(B) = \boxed{}$
Set *C* is the set of natural numbers less than 10 that are multiples of 4.	$\boxed{} = \left\{ 4, \boxed{} \right\}$	$n(C) = \boxed{}$
The universal set is all natural numbers less than 10.	$U =$	$\boxed{} \left(\boxed{} \right) = 9$

The following table identifies terms used to describe relationships among sets. Use sets A, B, C, and U from the previous table. You will supply the missing Venn diagrams in the Example column, including the referenced elements of the sets, as you complete steps B–I following.

Term	Notation	Venn Diagram	Example
Set C is a **subset** of set B if every element of C is also an element of B.	$C \subset B$	*Venn diagram: U containing oval B containing oval C.*	
The **intersection** of sets A and B is the set of all elements that are in both A *and* B.	$A \cap B$	*Venn diagram: U with overlapping ovals A and B, intersection shaded.* $A \cap B$ is the double-shaded region.	
The **union** of sets A and B is the set of all elements that are in A *or* B.	$A \cup B$	*Venn diagram: U with overlapping ovals A and B, entire region shaded.* $A \cup B$ is the entire shaded region.	
The **complement** of set A is the set of all elements in the universal set U that are *not* in A.	A^C or $\sim A$	*Venn diagram: U shaded with unshaded oval A.* A^C is the shaded region.	

(B) Since C is a subset of B, every element of set C, which consists of the numbers ____ and ____, is located not only in oval C, but also within oval B. Set B includes the elements of C as well as the additional elements ____ and ____, which are located in oval B outside of oval C. The universal set includes the elements of sets B and C as well as the additional elements ____, ____, ____, ____, and ____, which are located in region U outside of ovals B and C.

(C) In the first row of the table, draw the corresponding Venn diagram that includes the elements of B, C, and U. _____

(D) To determine the intersection of A and B, first define the elements of set A and set B separately, then identify all the elements found in both sets A *and* B.

$A = \{ \Box, \Box, \Box, \Box \}$

$B = \{ \Box, \Box, \Box, \Box \}$

$A \cap B = \{ \Box \}$

(E) In the second row of the table, draw the Venn diagram for $A \cap B$ that includes the elements of A, B, and U and the double-shaded intersection region. _____

(F) To determine the union of sets A and B, identify all the elements found in either set A or set B by combining all the elements of the two sets into the union set.

$A \cup B = \left\{ \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{} \right\}$

(G) In the third row of the table, draw the Venn diagram for $A \cup B$ that includes the elements of A, B, and U and the shaded union region. _____

(H) To determine the complement of set A, first identify the elements of set A and universal set U separately, then identify all the elements in the universal set that are *not* in set A.

$A = \left\{ \boxed{}, \boxed{}, \boxed{}, \boxed{} \right\}$

$U = \left\{ \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{} \right\}$

$A^C = \left\{ \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{} \right\}$

(I) In the fourth row of the table, draw the Venn diagram for A^C that includes the elements of A and U and the shaded region that represents the complement of A. _____

Reflect

1. **Draw Conclusions** Do sets always have an intersection that is not the empty set? Provide an example to support your conclusion.

_____

_____

🖋 Explain 1　Calculating Theoretical Probabilities

A *probability experiment* is an activity involving chance. Each repetition of the experiment is called a *trial* and each possible result of the experiment is termed an *outcome*. A set of outcomes is known as an *event,* and the set of all possible outcomes is called the *sample space.*

Probability measures how likely an event is to occur. An event that is impossible has a probability of 0, while an event that is certain has a probability of 1. All other events have a probability between 0 and 1. When all the outcomes of a probability experiment are equally likely, the **theoretical probability** of an event A in the sample space S is given by

$P(A) = \dfrac{\text{number of outcomes in the event}}{\text{number of outcomes in the sample space}} = \dfrac{n(A)}{n(S)}.$

Example 1 Calculate $P(A)$, $P(A \cup B)$, $P(A \cap B)$, and $P(A^C)$ for each situation.

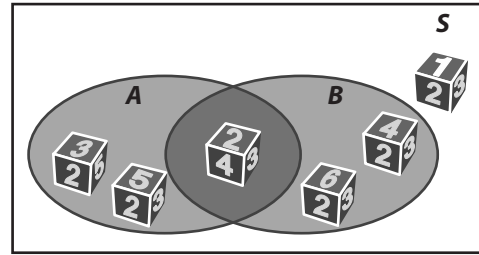(A) You roll a number cube. Event A is rolling a prime number. Event B is rolling an even number.

$S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$. $A = \{2, 3, 5\}$, so $n(A) = 3$.

So, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$.

$A \cup B = \{2, 3, 4, 5, 6\}$, so $n(A \cup B) = 5$. So, $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)} = \dfrac{5}{6}$.

$A \cap B = \{2\}$, so $n(A \cap B) = 1$. So, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{1}{6}$.

$A^C = \{1, 4, 6\}$, so $n(A^C) = 3$. So, $P(A^C) = \dfrac{n(A^C)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$.

(B) Your grocery basket contains one bag of each of the following items: oranges, green apples, green grapes, green broccoli, white cauliflower, orange carrots, and green spinach. You are getting ready to transfer your items from your cart to the conveyer belt for check-out. Event A is picking a bag containing a vegetable first. Event B is picking a bag containing a green food first. All bags have an equal chance of being picked first.

$S = \{\text{orange, apple, grape, broccoli, cauliflower, carrot, spinach}\}$, so $n(S) = \boxed{}$.

$A = \{\text{broccoli, cauliflower, }\underline{}, \underline{}\}$, so $n(A) = \boxed{}$. So $P(A) = \dfrac{n\left(\boxed{}\right)}{n\left(\boxed{}\right)} = \dfrac{\boxed{}}{\boxed{}}$.

$A \cup B = \{\text{broccoli, }\underline{}, \underline{}, \underline{}, \underline{}, \text{grape}\}$, so $n(A \cup B) = \boxed{}$.

$P(A \cup B) = \dfrac{\boxed{}\left(\boxed{}\ \boxed{}\right)}{\boxed{}\left(\boxed{}\right)} = \dfrac{\boxed{}}{\boxed{}}$

$A \cap B = \{\underline{}, \underline{}\}$, so $n(A \cap B) = \boxed{}$

$P(A \cap B) = \dfrac{\boxed{}\left(\boxed{}\ \boxed{}\right)}{\boxed{}\left(\boxed{}\right)} = \dfrac{\boxed{}}{\boxed{}}$

$P\left(\boxed{}\right) = \dfrac{\boxed{}\left(\boxed{}\right)}{\boxed{}\left(\boxed{}\right)} = \dfrac{\boxed{}}{\boxed{}}$

2. **Discussion** In Example 1B, which is greater, $P(A \cup B)$ or $P(A \cap B)$? Do you think this result is true in general? Explain.

_____

_____

_____

_____

Your Turn

The numbers 1 through 30 are written on slips of paper that are then placed in a hat. Students draw a slip to determine the order in which they will give an oral report. Event A is being one of the first 10 students to give their report. Event B is picking a multiple of 6. If you pick first, calculate each of the indicated probabilities.

3. $P(A)$

4. $P(A \cup B)$

5. $P(A \cap B)$

6. $P(A^c)$

⚙ Explain 2   Using the Complement of an Event

You may have noticed in the previous examples that the probability of an event occurring and the probability of the event not occurring (i.e., the probability of the complement of the event) have a sum of 1. This relationship can be useful when it is more convenient to calculate the probability of the complement of an event than it is to calculate the probability of the event.

Probabilities of an Event and Its Complement	
$P(A) + P(A^c) = 1$	The sum of the probability of an event and the probability of its complement is 1.
$P(A) = 1 - P(A^c)$	The probability of an event is 1 minus the probability of its complement.
$P(A^c) = 1 - P(A)$	The probability of the complement of an event is 1 minus the probability of the event.

Example 2 Use the complement to calculate the indicated probabilities.

(A) You roll a blue number cube and a white number cube at the same time. What is the probability that you do not roll doubles?

Step 1 Define the events. Let A be that you do not roll doubles and A^c that you do roll doubles.

Step 2 Make a diagram. A two-way table is one helpful way to identify all the possible outcomes in the sample space.

Blue Number Cube

White Number Cube	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Step 3 Determine $P(A^c)$. Since there are fewer outcomes for rolling doubles, it is more convenient to determine the probability of rolling doubles, which is $P(A^c)$. To determine $n(A^c)$, draw a loop around the outcomes in the table that correspond to A^c and then calculate $P(A^c)$.

$$P(A^c) = \frac{n(A^c)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Step 4 Determine $P(A)$. Use the relationship between the probability of an event and its complement to determine $P(A)$.

$$P(A) = 1 - P(A^c) = 1 - \frac{1}{6} = \frac{5}{6}$$

So, the probability of not rolling doubles is $\frac{5}{6}$.

(B) One pile of cards contains the numbers 2 through 6 in red hearts. A second pile of cards contains the numbers 4 through 8 in black spades. Each pile of cards has been randomly shuffled. If one card from each pile is chosen at the same time, what is the probability that the sum will be less than 12?

Step 1 Define the events. Let A be the event that the sum is less than 12 and A^c be the event that _____.

Step 2 Make a diagram. Complete the table to show all the outcomes in the sample space.

Step 3 Determine $P(A^c)$. Circle the outcomes in the table that correspond to A^c, then determine $P(A^c)$.

$$P(A^c) = \frac{\boxed{}\left(\boxed{}\right)}{\boxed{}\left(\boxed{}\right)} = \frac{\boxed{}}{\boxed{}}$$

Black Spades ♠	Red Hearts ♥				
	2	3	4	5	6
4	4+2	4+3			
5	5+2				
6					
7					
8					

Step 4 Determine P(A). Use the relationship between the probability of an event and its complement to determine $P(A^c)$.

$$P(A) = \boxed{} - \boxed{}\left(\boxed{}\right) = \boxed{} - \frac{\boxed{}}{\boxed{}} = \frac{\boxed{}}{\boxed{}}$$

So, the probability that the sum of the two cards is _____ is $\dfrac{\boxed{}}{\boxed{}}$.

7. Describe a different way to calculate the probability that the sum of the two cards will be less than 12.

_____

_____

One bag of marbles contains two red, one yellow, one green, and one blue marble. Another bag contains one marble of each of the same four colors. One marble from each bag is chosen at the same time. Use the complement to calculate the indicated probabilities.

8. Probability of selecting two different colors

9. Probability of not selecting a yellow marble

💬 Elaborate

10. Can a subset of A contain elements of A^C? Why or why not?

_____

_____

_____

11. For any set A, what does $A \cap \varnothing$ equal? What does $A \cup \varnothing$ equal? Explain.

_____

_____

_____

12. Essential Question Check-In How do the terms *set*, *element*, and *universal set* correlate to the terms used to calculate theoretical probability?

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Set A is the set of factors of 12, set B is the set of even natural numbers less than 13, set C is the set of odd natural numbers less than 13, and set D is the set of even natural numbers less than 7. The universal set for these questions is the set of natural numbers less than 13.

So, $A = \{1, 2, 3, 4, 6, 12\}$, $B = \{2, 4, 6, 8, 10, 12\}$, $C = \{1, 3, 5, 7, 9, 11\}$, $D = \{2, 4, 6\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Answer each question.

1. Is $D \subset A$? Explain why or why not.

2. Is $B \subset A$? Explain why or why not.

3. What is $A \cap B$?

4. What is $A \cap C$?

5. What is $A \cup B$?

6. What is $A \cup C$?

7. What is A^C?

8. What is B^C?

You have a set of 10 cards numbered 1 to 10. You choose a card at random. Event A is choosing a number less than 7. Event B is choosing an odd number. Calculate the probability.

9. $P(A)$

10. $P(B)$

11. $P(A \cup B)$

12. $P(A \cap B)$

13. $P(A^C)$

14. $P(B^C)$

Use the complement of the event to find the probability.

15. You roll a 6-sided number cube. What is the probability that you do not roll a 2?

16. You choose a card at random from a standard deck of cards. What is the probability that you do not choose a red king?

17. You spin the spinner shown. The spinner is divided into 12 equal sectors. What is the probability of not spinning a 2?
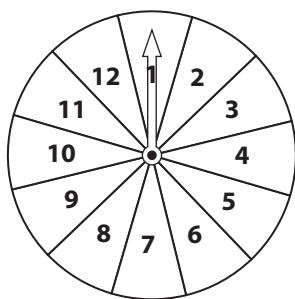
18. A bag contains 2 red, 5 blue, and 3 green balls. A ball is chosen at random. What is the probability of not choosing a red ball?

19. Cards numbered 1–12 are placed in a bag. A ball is chosen at random. What is the probability of not choosing a number less than 5?

20. Slips of paper numbered 1–20 are folded and placed into a hat, and then a slip of paper is drawn at random. What is the probability the slip drawn has a number which is not a multiple of 4 or 5?

21. You are going to roll two number cubes, a white number cube and a red number cube, and find the sum of the two numbers that come up.

    a. What is the probability that the sum will be 6?

    b. What is the probability that the sum will not be 6?

22. You have cards with the letters A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P. Event U is choosing the cards A, B, C or D. Event V is choosing a vowel. Event W is choosing a letter in the word "APPLE". Find $P(U \cap V \cap W)$.

A standard deck of cards has 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, ace) in each of 4 suits (hearts, clubs, diamonds, spades). The hearts and diamonds cards are red. The clubs and spades cards are black. Answer each question.

23. You choose a card from a standard deck of cards at random. What is the probability that you do not choose an ace? Explain.

24. You choose a card from a standard deck of cards at random. What is the probability that you do not choose a club? Explain.

25. You choose a card from a standard deck of cards at random. Event A is choosing a red card. Event B is choosing an even number. Event C is choosing a black card. Find $P(A \cap B \cap C)$. Explain.

26. You are selecting a card at random from a standard deck of cards. Match each event with the correct probability. Indicate a match by writing the letter of the event on the line in front of the corresponding probability.

A. Picking a card that is both red and a heart.   _____ $\dfrac{1}{52}$

B. Picking a card that is both a heart and an ace.   _____ $\dfrac{1}{4}$

C. Picking a card that is not both a heart and an ace.   _____ $\dfrac{51}{52}$

27. **Critique Reasoning** A bag contains white tiles, black tiles, and gray tiles. Someone is going to choose a tile at random. $P(W)$, the probability of choosing a white tile, is $\frac{1}{4}$. A student claims that the probability of choosing a black tile, $P(B)$, is $\frac{3}{4}$ since $P(B) = 1 - P(W) = 1 - \frac{1}{4} = \frac{3}{4}$. Do you agree? Explain.

28. **Communicate Mathematical Ideas** A bag contains 5 red marbles and 10 blue marbles. You are going to choose a marble at random. Event A is choosing a red marble. Event B is choosing a blue marble. What is $P(A \cap B)$? Explain.

29. **Critical Thinking** Jeffery states that for a sample space S where all outcomes are equally likely, $0 \le P(A) \le 1$ for any subset A of S. Create an argument that will justify his statement or state a counterexample.

Lesson Performance Task

For the sets you've worked with in this lesson, membership in a set is binary: Either something belongs to the set or it doesn't. For instance, 5 is an element of the set of odd numbers, but 6 isn't

In 1965, Lofti Zadeh developed the idea of "fuzzy" sets to deal with sets for which membership is not binary. He defined a *degree* of membership that can vary from 0 to 1. For instance, a membership function $m_L(w)$ for the set L of large dogs where the degree of membership m is determined by the weight w of a dog might be defined as follows:

- A dog is a full member of the set L if it weighs 80 pounds or more. This can be written as $m_L(w) = 1$ for $w \geq 80$.

- A dog is not a member of the set L if it weighs 60 pounds or less. This can be written as $m_L(w) = 0$ for $w \leq 60$.

- A dog is a partial member of the set L if it weighs between 60 and 80 pounds. This can be written as $0 < m_L(w) < 1$ for $60 < w < 80$.

The "large dogs" portion of the graph shown displays the membership criteria listed above. Note that the graph shows only values of $m(w)$ that are positive.
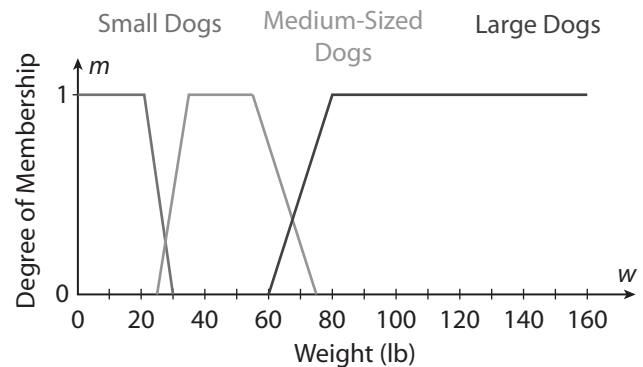
1. Using the graph, give the approximate weights for which a dog is considered a full member, a partial member, and not a member of the set S of small dogs.
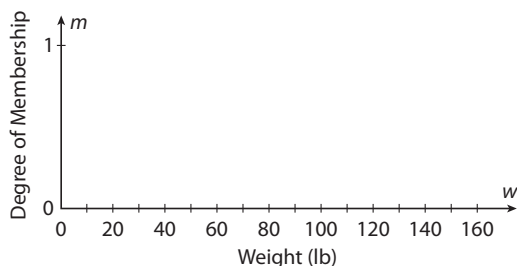
2. The union of two "fuzzy" sets A and B is given by the membership rule $m_{A \cup B}(x) = \text{maximum}(m_A(x), m_B(x))$. So, for a dog of a given size, the degree of its membership in the set of small or medium-sized dogs $(S \cup M)$ is the greater of its degree of membership in the set of small dogs and its degree of membership in the set of medium-sized dogs.

   The intersection of A and B is given by the membership rule $m_{A \cap B}(x) = \text{minimum}(m_A(x), m_B(x))$. So, for a dog of a given size, the degree of its membership in the set of dogs that are both small and medium-sized $(S \cap M)$ is the lesser of its degree of membership in the set of small dogs and its degree of membership in the set of medium-sized dogs.

   Using the graph above and letting S be the set of small dogs, M be the set of medium-sized dogs, and L be the set of large dogs, draw the graph of each set.

   a. $S \cup M$

   Approximately size shown here

   b. $M \cap L$

   Approximately size shown here