Name ______________________ Class ______________ Date ________

17.1 Angles of Rotation and Radian Measure

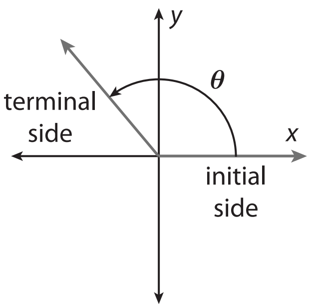
Essential Question: What is the relationship between the unit circle and radian measure?

Resource Locker

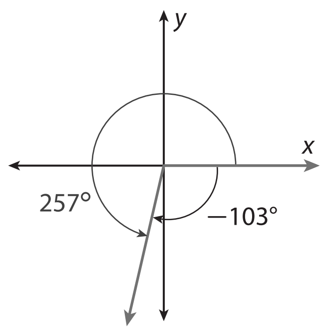
Explore 1 Drawing Angles of Rotation and Finding Coterminal Angles

In trigonometry, an **angle of rotation** is an angle formed by the starting and ending positions of a ray that rotates about its endpoint. The angle is in *standard position* in a coordinate plane when the starting position of the ray, or *initial side* of the angle, is on the positive x-axis and has its endpoint at the origin. To show the amount and direction of rotation, a curved arrow is drawn to the ending position of the ray, or *terminal side* of the angle.

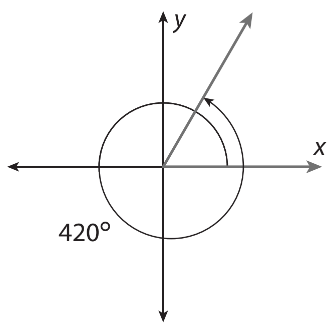
In geometry, you were accustomed to working with angles having measures between 0° and 180°. In trigonometry, angles can have measures greater than 180° and even less than 0°. To see why, think in terms of revolutions, or complete circular motions. Let θ be an angle of rotation in standard position.



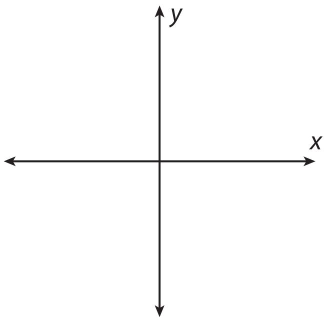
- If the rotation for an angle θ is less than 1 revolution in a counterclockwise direction, then the measure of θ is between 0° and 360°. An angle of rotation measured *clockwise* from standard position has a *negative* angle measure. **Coterminal angles** are angles that share the same terminal side. For example, the angles with measures of 257° and −103° are coterminal, as shown.



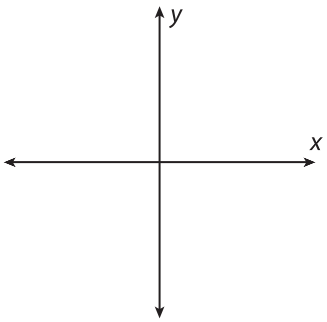
- If the rotation for θ is more than 1 revolution but less than 2 revolutions in a counterclockwise direction, then the measure of θ is between 360° and 720°, as shown. Because you can have any number of revolutions with an angle of rotation, there is a counterclockwise angle of rotation corresponding to any positive real number and a clockwise angle of rotation corresponding to any negative real number.



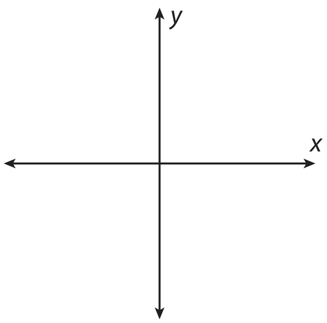
Ⓐ Draw an angle of rotation of 310°. In what quadrant is the terminal side of the angle?



Ⓑ On the same graph from the previous step, draw a positive coterminal angle. What is the angle measure of your angle?



Ⓒ On the same graph from the previous two steps, draw a negative coterminal angle. What is the angle measure of your angle?



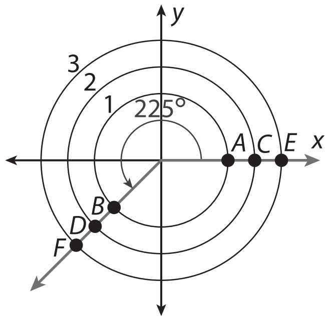
Reflect

1. Is the measure of an angle of rotation in standard position completely determined by the position of its terminal side? Explain.

2. Find the measure between 720° and 1080° of an angle that is coterminal with an angle that has a measure of −30°. In addition, describe a general method for finding the measure of any angle that is coterminal with a given angle.

Explore 2 Understanding Radian Measure

The diagram shows three circles centered at the origin. The arcs that are on the circle between the initial and terminal sides of the 225° central angle are called *intercepted arcs*.



$\overset{\frown}{AB}$ is on a circle with radius 1 unit.

$\overset{\frown}{CD}$ is on a circle with radius 2 units.

$\overset{\frown}{EF}$ is on a circle with radius 3 units.

Notice that the intercepted arcs have different lengths, although they are intercepted by the same central angle of 225°. You will now explore how these arc lengths are related to the angle.

(A) The angle of rotation is $\square$ degrees counterclockwise.

There are $\square$ degrees in a circle.

225° represents $\frac{\square}{\square}$ of the total number of degrees in a circle.

So, the length of each intercepted arc is $\frac{\square}{\square}$ of the total circumference of the circle that it lies on.

(B) Complete the table. To find the length of the intercepted arc, use the fraction you found in the previous step. Give all answers in terms of π.

Radius, r	Circumference, C $(C = 2\pi r)$	Length of Intercepted Arc, s	Ratio of Arc Length to Radius, $\frac{s}{r}$
1			
2			
3			

Reflect

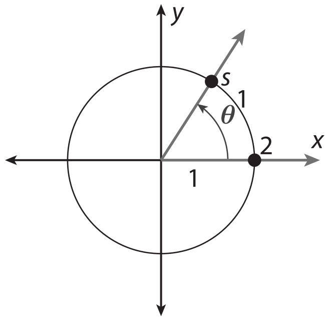
3. What do you notice about the ratios $\frac{s}{r}$ in the fourth column of the table?

4. When the ratios of the values of a variable y to the corresponding values of another variable x all equal a constant k, y is said to be *proportional* to x, and the constant k is called the *constant of proportionality*. Because $\frac{y}{x} = k$, you can solve for y to get $y = kx$. In the case of the arcs that are intercepted by a 225° angle, is the arc length s proportional to the radius r? If so, what is the constant of proportionality, and what equation gives s in terms of r?

5. Suppose that the central angle is 270° instead of 225°. Would the arc length s still be proportional to the radius r? If so, would the constant of proportionality still be the same? Explain.

Explain 1 Converting Between Degree Measure and Radian Measure

For a central angle θ that intercepts an arc of length s on a circle with radius r, the **radian measure** of the angle is the ratio $\theta = \frac{s}{r}$. In particular, on a *unit circle*, a circle centered at the origin with a radius of 1 unit, $\theta = s$. So, 1 *radian* is the angle that intercepts an arc of length 1 on a unit circle, as shown.



$\theta = 1$ radian

Recall that there are 360° in a full circle. Since the circumference of a circle of radius r is $s = 2\pi r$, the number of radians in a full circle is $\frac{2\pi r}{r} = 2\pi$. Therefore, $360° = 2\pi$ radians. So, $1° = \frac{2\pi}{360} = \frac{\pi}{180}$ radians and 1 radian $= \frac{360}{2\pi} = \frac{180}{\pi}$ degrees. This result is summed up in the following table.

CONVERTING DEGREES TO RADIANS	CONVERTING RADIANS TO DEGREES
Multiply the number of degrees by $\left(\frac{\pi \text{ radians}}{180°}\right)$.	Multiply the number of radians by $\left(\frac{180°}{\pi \text{ radians}}\right)$.

Example 1 **Convert each measure from degrees to radians or from radians to degrees.**

(A)

Degree measure	Radian measure
20°	$\frac{\pi}{180°} \cdot 20° = \frac{\pi}{9}$
315°	$\frac{\pi}{180°} \cdot 315° = \frac{7\pi}{4}$
600°	$\frac{\pi}{180°} \cdot 600° = \frac{10\pi}{3}$
−60°	$\frac{\pi}{180°} \cdot (-60°) = -\frac{\pi}{3}$
−540°	$\frac{\pi}{180°} \cdot (-540°) = -3\pi$

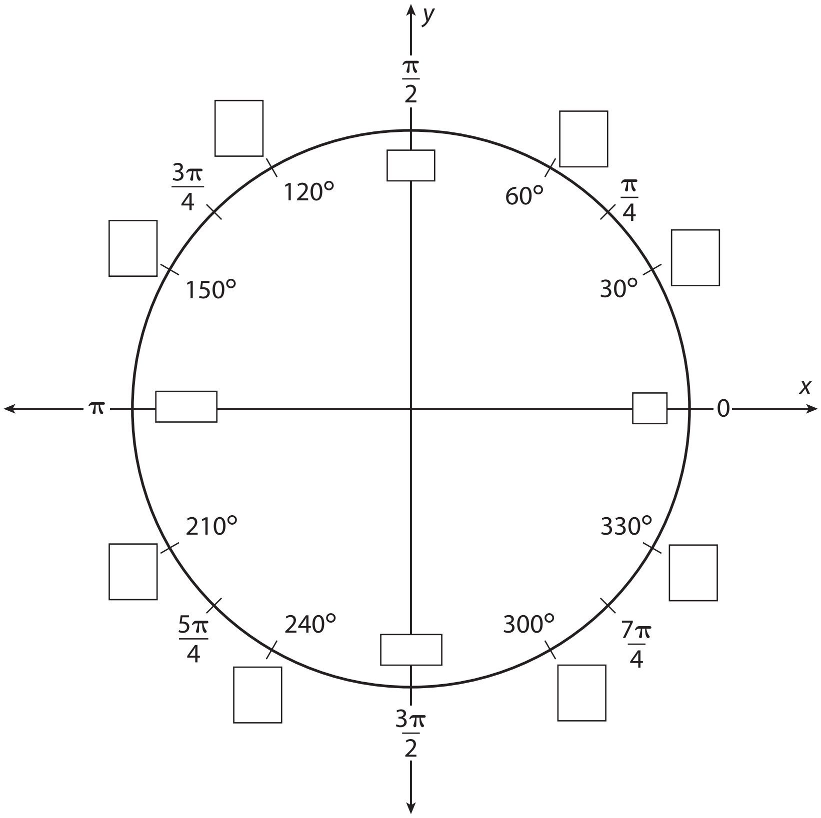
(B)

Radian measure	Degree measure
$\frac{\pi}{8}$	$\frac{180°}{\pi} \cdot \frac{\pi}{8} = \boxed{}$
$\frac{4\pi}{3}$	$\boxed{} \cdot \frac{4\pi}{3} = \boxed{}$
$\frac{9\pi}{2}$	
$-\frac{7\pi}{12}$	
$-\frac{13\pi}{6}$	

Reflect

6. Which is larger, a degree or a radian? Explain.

7. The unit circle below shows the measures of angles of rotation that are commonly used in trigonometry, with radian measures outside the circle and degree measures inside the circle. Provide the missing measures.



Your Turn

Convert each measure from degrees to radians or from radians to degrees.

8. $-495°$

9. $\frac{13\pi}{12}$

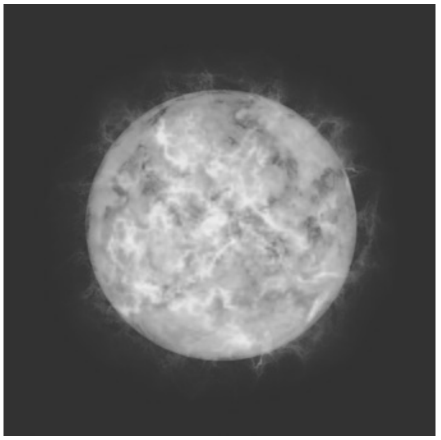
Explain 2 Solving a Real-World Problem Involving Arc Length

As you saw in the first Explain, for a central angle θ in radian measure, $\theta = \frac{s}{r}$ where s is the intercepted arc length. Multiplying both sides of the equation by r gives the arc length formula for a circle:

Arc Length Formula
For a circle of radius r, the arc length s intercepted by a central angle θ (measured in radians) is given by the following formula. $s = r\theta$

Many problems involving arc length also involve *angular velocity*, which is the angle measure through which an object turns in a given time interval. For example, the second hand of a clock has an angular velocity of 360° per minute, or 6° per second. Angular velocity may also be expressed in radians per unit of time. This makes finding the arc length traversed in an amount of time especially easy by using the arc length formula.

(A) **The Sun** A point on the Sun's equator makes a full revolution once every 25.38 days. The Sun has a radius of about 432,200 miles at its equator. What is the angular velocity in radians per hour of a point on the Sun's equator? What distance around the Sun's axis does the point travel in one hour? How does this compare with the distance of about 1038 miles traveled by a point on Earth's equator in an hour?



One revolution is 2π radians. The angular velocity in radians per day is $\frac{2\pi}{25.38}$. Convert this to radians per hour.

$$\frac{2\pi \text{ radians}}{25.38 \text{ days}} \cdot \frac{1 \text{ day}}{24 \text{ hours}} = \frac{2\pi \text{ radians}}{25.38(24) \text{ hours}}$$

$$\approx 0.01032 \text{ radians/h}$$

The distance the point travels in an hour is the arc length it traverses in an hour.

$$s = r\theta$$

$$= 432{,}200(0.01032)$$

$$\approx 4460$$

The point travels about 4460 miles around the Sun's axis in an hour. This is more than 4 times farther than a point on Earth's equator travels in the same time.

(B) **The Earth** Earth's equator is at a latitude of 0°. The Arctic circle is at a latitude of 66.52°N. The diameter of the equator is 7926 miles. The diameter of the Arctic circle is 3150 miles.

a. Find the angular velocity in degrees per minute of a point on the equator and of a point on the Arctic circle.

b. How far does a point on the Equator travel in 15 minutes?

c. How long will it take a point on the Arctic circle to travel this distance?

a. Every point on Earth completes 1 revolution of ______ degrees each 24 hours, so the angular velocities of the points will be the same. Convert the angular velocity to degrees per minute.

$$\frac{\boxed{}^\circ}{\boxed{}\text{ h}} \cdot \frac{1\text{ h}}{60\text{ min}} = \frac{\boxed{}^\circ}{24(60)\text{ min}} = \boxed{}^\circ/\text{min}$$

The angular velocity is $\boxed{}^\circ$/min.

b. Multiply the time by the angular velocity to find the angle through which a point rotates in 15 minutes.

$\boxed{}$ min $\cdot$ 0.25°/min $=$ $\boxed{}^\circ$

Write a proportion to find the distance to the nearest tenth that this represents at the equator, where Earth's circumference is $\boxed{}$ · 7926 miles.

$$\frac{3.75^\circ}{\boxed{}^\circ} = \frac{x\text{ mi}}{\boxed{} \cdot 7926\text{ mi}}$$

$$x = \frac{3.75\pi(7926)}{360}$$

$$x \approx \boxed{}$$

A point at the equator travels about ______ miles in 15 minutes.

c. Write a proportion to find the angle of rotation to the nearest thousandth required to move a point 259.4 miles on the Arctic circle, where the circumference is $\boxed{}$ · 3150 miles.

$$\frac{259.4\text{ mi}}{\boxed{} \cdot 3150\text{ mi}} = \frac{x^\circ}{\boxed{}^\circ}$$

$$x = \frac{259.4(360)}{3150\pi}$$

$$x \approx \boxed{}$$

Use the angular velocity to find the time t to the nearest hundredth required for a point on the Arctic circle to move through an angle of rotation of 9.437°.

$$\left(\boxed{}^\circ/\text{min}\right)(t\text{ min}) = 9.437^\circ$$

$$t \approx \boxed{}$$

It takes about ______ minutes for a point on the Arctic circle to travel the same distance that a point on the equator travels in 15 minutes.

Reflect

10. How does using an angle of rotation to find the length of the arc on a circle intercepted by the angle differ when degrees are used from when radians are used?

Your Turn

11. **Astronomy** A neutron star (an incredibly dense collapsed star) in the Sagittarius Galaxy has a radius of 10 miles and completes a full revolution every 0.0014 seconds. Find the angular velocity of the star in radians per second, then use this velocity to determine how far a point on the equator of the star travels each second. How does this compare to the speed of light (about 186,000 mi/sec)?

12. **Geography** The northeastern corner of Maine is due north of the southern tip of South America in Chile. The difference in latitude between the locations is 103°. Using both degree measure and radian measure, and a north-south circumference of Earth of 24,860 miles, find the distance between the two locations.

Elaborate

13. Given the measure of two angles of rotation, how can you determine whether they are coterminal without actually drawing the angles?

14. What is the conversion factor to go from degrees to radians? What is the conversion factor to go from radians to degrees? How are the conversion factors related?

15. **Essential Question Check-In** An angle of rotation in standard position intercepts an arc of length 1 on the unit circle. What is the radian measure of the angle of rotation?

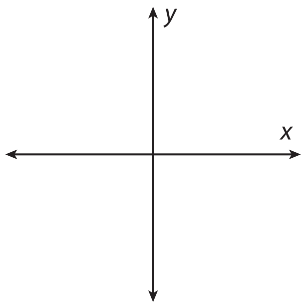
Evaluate: Homework and Practice

Personal Math Trainer

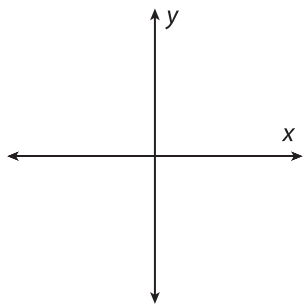
- Online Homework
- Hints and Help
- Extra Practice

Draw the indicated angle of rotation in standard position.

1. A positive angle coterminal to 130°



2. A negative angle coterminal to 130°



For each angle, find the nearest two positive coterminal angles and the nearest two negative coterminal angles.

3. 84°

4. 420°

5. $-\frac{\pi}{3}$

6. $\frac{5\pi}{2}$

Convert each measure from degrees to radians or from radians to degrees.

7. 70°

8. −270°

9. −945°

10. 2160°

11. $\frac{33\pi}{18}$

12. $\frac{11\pi}{4}$

13. $-\frac{5\pi}{3}$

14. $-\frac{7\pi}{2}$

15. **Geography** A student in the United States has an friend overseas with whom she corresponds by computer. The foreign student says, "If you write the latitude and longitude of my school in radians instead of degrees, you get the coordinates 0.6227 radians north latitude and 2.438 radians east longitude". Convert the coordinates back to degrees. Then use a globe, map, or app to identify the city.

16. **Geography** If a ship sailed due south from Iceland to Antarctica, it would sail through an angle of rotation of about 140° around Earth's center. Find this measure in radians. Then, using 3960 miles for Earth's radius, find how far the ship would travel.

17. **Geography** Acapulco, Mexico and Hyderabad, India both lie at 17° north latitude, and lie very nearly halfway around the world from each other in an east-west direction. The radius of Earth at a latitude of 17° is about 3790 miles. Suppose that you could fly from Acapulco directly west to Hyderabad or fly directly north to Hyderabad. Which way would be shorter, and by how much? Use 3960 miles for Earth's radius. (Hint: To fly directly north, you would go from 17° north latitude to 90° north latitude, and then back down to 17° north latitude.)

18. **Planetary Exploration** "Opportunity" and "Phoenix" are two of the robotic explorers on Mars. Opportunity landed at 2° south latitude, where Mars' radius is about 2110 miles. Phoenix landed at 68° north latitude, where Mars' radius is about 790 miles. Mars rotates on its axis once every 24.6 Earth-hours. How far does each explorer travel as Mars rotates by 1 radian? How many hours does it take Mars to rotate 1 radian? Using this answer, how fast is each explorer traveling around Mars' axis in miles per hour?

19. **Earth's Rotation** The 40th parallel of north latitude runs across the United States through Philadelphia, Indianapolis, and Denver. At this latitude, Earth's radius is about 3030 miles. The earth rotates with an angular velocity of $\frac{\pi}{12}$ radians (or 15°) per hour toward the east. If a jet flies due west with the same angular velocity relative to the ground at the equinox, the Sun as viewed from the jet will stop in the sky. How fast in miles per hour would the jet have to travel west at the 40th parallel for this to happen?

20. **Our Galaxy** It is about 30,000 light years from our solar system to the center of the Milky Way Galaxy. The solar system revolves around the center of the Milky Way with an angular velocity of about 2.6×10^{-8} radians per year.

a. What distance does the solar system travel in its orbit each year?

b. Given that a light year is about 5.88×10^{12} miles, how fast is the solar system circling the center of the galaxy in miles per hour?

21. **Driving** A windshield wiper blade turns through an angle of 135°. The bottom of the blade traces an arc with a 9-inch radius. The top of the blade traces an arc with a 23-inch radius. To the nearest inch, how much longer is the top arc than the bottom arc?

22. **Cycling** You are riding your bicycle, which has tires with a 30-inch diameter, at a steady 15 miles per hour. What is the angular velocity of a point on the outside of the tire in radians per second?

23. Select all angles that are coterminal with an angle of rotation of 300°.

A. $-420°$

B. $2100°$

C. $-900°$

D. $-\frac{\pi}{3}$ radians

E. $\frac{23\pi}{3}$ radians

F. $-\frac{7\pi}{3}$ radians

H.O.T. Focus on Higher Order Thinking

24. **Explain the Error** Lisa was told that a portion of the restaurant on the Space Needle in Seattle rotates at a rate of 8 radians per hour. When asked to find the distance through which she would travel if she sat at a table 40 feet from the center of rotation for a meal lasting 2 hours, she produced the following result:

$\theta = \frac{s}{r}$

$8 = \frac{s}{40}$

$320 = s$ The distance is 320 feet.

25. Represent Real-World Problems The minute hand on a clock has an angular velocity of 2π radians/hour, while the hour hand has an angular velocity of $\frac{\pi}{6}$ radians/hour. At 12:00, the hour and second hands both point straight up. The two hands will next come back together sometime after 1:00. At what exact time will this happen? (Hint: You want to find the next time when the angle of rotation made by the hour hand is coterminal with the angle made by the minute hand after it has first completed one full revolution.)

26. Critical Thinking Write a single rational expression that can be used to represent all angles that are coterminal with an angle of $\frac{5\pi}{8}$ radians.

27. Extension You know that the length s of the arc intercepted on a circle of radius r by an angle of rotation of θ radians is $s = r\theta$. Find an expression for the area of the sector of the circle with radius r that has a central angle of θ radians. Explain your reasoning.

Lesson Performance Task

At a space exploration center, astronauts are training on a human centrifuge that has a diameter of 70 feet.

a. The centrifuge makes 72 complete revolutions in 2 minutes. What is the angular velocity of the centrifuge in radians per second? What distance does an astronaut travel around the center each second?

b. Acceleration is the rate of change of velocity with time. An object moving at a constant velocity v in circular motion with a radius of r has an acceleration a of $a = \frac{v^2}{r}$. What is the astronaut's acceleration? (Note that the acceleration will have units of feet per second *squared*.)

c. One "g" is the acceleration caused on Earth's surface by gravity. This acceleration is what gives you your weight. Some roller coasters can produce an acceleration in a tight loop of 5 or even 6 g's. Earth's gravity produces an acceleration of 32 ft/s^2. How many g's is the astronaut experiencing in the centrifuge?