Name ______________________ Class ______________ Date __________

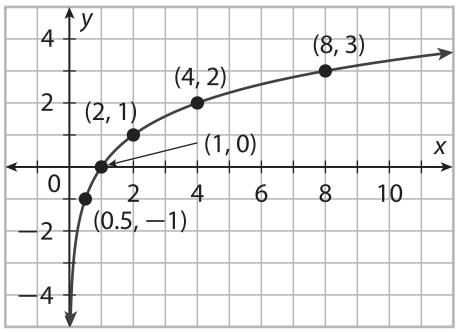
15.2 Graphing Logarithmic Functions

Resource Locker

Essential Question: How is the graph of $g(x) = a\log_b(x-h) + k$ where $b > 0$ and $b \neq 1$ related to the graph of $f(x) = \log_b x$?

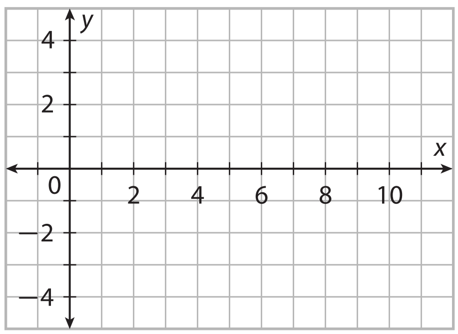
Explore 1 Graphing and Analyzing Parent Logarithmic Functions

The graph of the logarithmic function $f(x) = \log_2 x$, which you analyzed in the previous lesson, is shown. In this Explore, you'll graph and analyze other basic logarithmic functions.



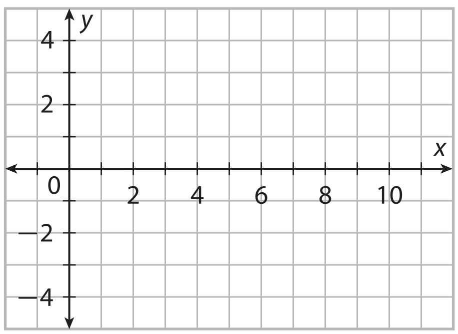
(A) Complete the table for the function $f(x) = \log x$. (Remember that when the base of a logarithmic function is not specified, it is understood to be 10.) Then plot and label the ordered pairs from the table and draw a smooth curve through the points to obtain the graph of the function.

x	$f(x) = \log x$
0.1	
1	
10	



(B) Complete the table for the function $f(x) = \ln x$. (Remember that the base of this function is e). Then plot and label the ordered pairs from the table and draw a smooth curve through the points to obtain the graph of the function.

x	$f(x) = \ln x$
$\frac{1}{e} \approx 0.368$	
1	
$e \approx 2.72$	
$e^2 \approx 7.39$	



Ⓒ Analyze the two graphs from Steps A and B, and then complete the table.

Function	$f(x) = \log_2(x)$	$f(x) = \log x$	$f(x) = \ln x$
Domain	$\{x \mid x > 0\}$		
Range	$\{y \mid 0 < y < \infty\}$		
End behavior	As $x \to +\infty$, $f(x) \to +\infty$. As $x \to 0^+$, $f(x) \to -\infty$.		
Vertical and horizontal asymptotes	Vertical asymptote at $x = 0$; no horizontal asymptote		
Intervals where increasing or decreasing	Increasing throughout its domain		
Intercepts	x-intercept at $(1, 0)$; no y-intercepts		
Intervals where positive or negative	Positive on $(1, +\infty)$; negative on $(0, 1)$		

Reflect

1. What similarities do you notice about all logarithmic functions of the form $f(x) = \log_b x$ where $b > 1$? What differences do you notice?

Explain 1 Graphing Combined Transformations of $f(x) = \log_b x$ Where $b > 1$

When graphing transformations of $f(x) = \log_b x$ where $b > 1$, it helps to consider the effect of the transformations on the following features of the graph of $f(x)$: the vertical asymptote, $x = 0$, and two reference points, $(1, 0)$ and $(b, 1)$. The table lists these features as well as the corresponding features of the graph of $g(x) = a \log_b (x - h) + k$.

Function	$f(x) = \log_b x$	$g(x) = a \log_b (x - h) + k$
Asymptote	$x = 0$	$x = h$
Reference point	$(1, 0)$	$(1 + h, k)$
Reference point	$(b, 1)$	$(b + h, a + k)$

Example 1 **Identify the transformations of the graph of $f(x) = \log_b x$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptote $x = 0$ and to the reference points $(1, 0)$ and $(b, 1)$. Also state the domain and range of $g(x)$ using set notation.**

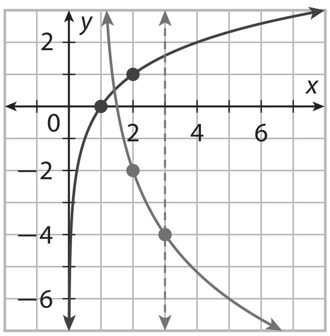
(A) $g(x) = -2 \log_2 (x - 1) - 2$

The transformations of the graph of $f(x) = \log_2 x$ that produce the graph of $g(x)$ are as follows:

- a vertical stretch by a factor of 2
- a reflection across the x-axis
- a translation of 1 unit to the right and 2 units down

Note that the translation of 1 unit to the right affects only the x-coordinates of points on the graph of $f(x)$, while the vertical stretch by a factor of 2, the reflection across the x-axis, and the translation of 2 units down affect only the y-coordinates.

Function	$f(x) = \log_2 x$	$g(x) = -2 \log_2 (x - 1) - 2$
Asymptote	$x = 0$	$x = 1$
Reference point	$(1, 0)$	$(1 + 1, -2(0) - 2) = (2, -2)$
Reference point	$(2, 1)$	$(2 + 1, -2(1) - 2) = (3, -4)$



Domain: $\{x \mid x > 1\}$

Range: $\{y \mid \infty < y < +\infty\}$

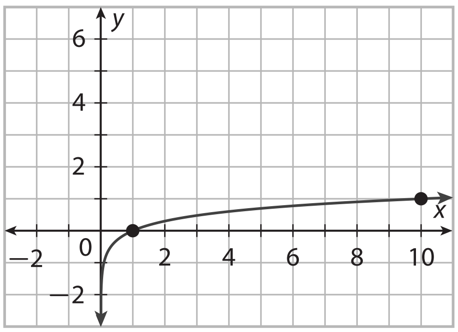
B) $g(x) = 2\log(x + 2) + 4$

The transformations of the graph of $f(x) = \log x$ that produce the graph of $g(x)$ are as follows:

- a vertical stretch by a factor of 2
- a translation of 2 units to the left and 4 units up

Note that the translation of 2 units to the left affects only the x-coordinates of points on the graph of $f(x)$, while the vertical stretch by a factor of 2 and the translation of 4 units up affect only the y-coordinates.

Function	$f(x) = \log x$	$g(x) = 2\log(x + 2) + 4$
Asymptote	$x = 0$	$x = \square$
Reference point	$(1, 0)$	$\left(\square - 2, 2\left(\square\right) + 4\right) = \left(\square, \square\right)$
Reference point	$(10, 1)$	$\left(\square - 2, 2\left(\square\right) + 4\right) = \left(\square, \square\right)$



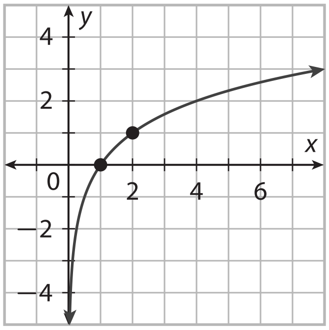
Domain: $\left\{x \mid x > \square\right\}$

Range: $\left\{y \mid -\infty < y < \square\right\}$

Your Turn

Identify the transformations of the graph of $f(x) = \log_b x$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptote $x = 0$ and to the reference points $(1, 0)$ and $(b, 1)$. Also state the domain and range of $g(x)$ using set notation.

2. $g(x) = \frac{1}{2}\log_2(x + 1) + 2$



Explain 2 Writing, Graphing, and Analyzing a Logarithmic Model

You can obtain a logarithmic model for real-world data either by performing logarithmic regression on the data or by finding the inverse of an exponential model if one is available.

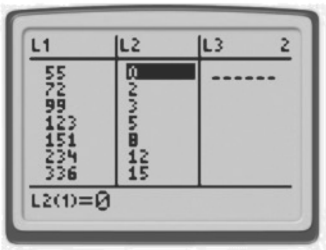
Example 2 **A biologist studied a population of foxes in a forest preserve over a period of time. The table gives the data that the biologist collected.**

Years Since Study Began	Fox Population
0	55
2	72
3	99
5	123
8	151
12	234
15	336
18	475

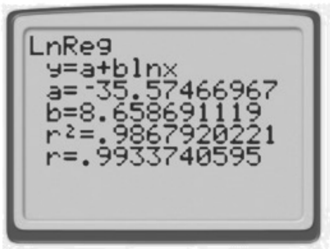
From the data, the biologist obtained the exponential model $P = 62(1.12)^t$ where P is the fox population at time t (in years since the study began). The biologist is interested in having a model that gives the time it takes the fox population to reach a certain level.

(A) One way to obtain the model that the biologist wants is to perform logarithmic regression on a graphing calculator using the data set but with the variables switched (that is, the fox population is the independent variable and time is the dependent variable). After obtaining the logarithmic regression model, graph it on a scatter plot of the data. Analyze the model in terms of whether it is increasing or decreasing as well as its average rate of change from $P = 100$ to $P = 200$, from $P = 200$ to $P = 300$, and from $P = 300$ to $P = 400$. Do the model's average rates of change increase, decrease, or stay the same? What does this mean for the fox population?

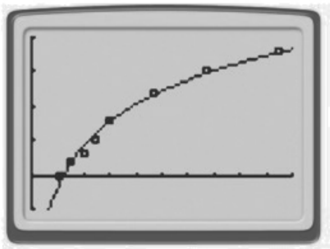
Using a graphing calculator, enter the population data into one list (L1) and the time data into another list (L2).



Perform logarithmic regression by pressing the LIST key, choosing the **CALC** menu, and selecting **9:LnReg**. Note that the calculator's regression model is a natural logarithmic function.



So, the model is $t = -35.6 + 8.66 \ln P$. Graphing this model on a scatter plot of the data visually confirms that the model is a good fit for the data.



From the graph, you can see that the function is increasing. To find the model's average rates of change, divide the change in t (the dependent variable) by the change in P (the independent variable):

Average rate of change $= \dfrac{t_2 - t_1}{P_2 - P_1}$

Population	Number of Years to Reach That Population	Average Rate of Change
100	$t = -35.6 + 8.66 \ln 100 \approx 4.3$	
200	$t = -35.6 + 8.66 \ln 200 \approx 10.3$	$\dfrac{10.3 - 4.3}{200 - 100} = \dfrac{6.0}{100} = 0.060$
300	$t = -35.6 + 8.66 \ln 300 \approx 13.8$	$\dfrac{13.8 - 10.3}{300 - 200} = \dfrac{3.5}{100} = 0.035$
400	$t = -35.6 + 8.66 \ln 400 \approx 16.3$	$\dfrac{16.3 - 13.8}{400 - 300} = \dfrac{2.5}{100} = 0.025$

The model's average rates of change are decreasing. This means that as the fox population grows, it takes less time for the population to increase by another 100 foxes.

B Another way to obtain the model that the biologist wants is to find the inverse of the exponential model. Find the inverse model and compare it with the logarithmic regression model.

In order to compare the inverse of the biologist's model, $P = 62(1.12)^t$, with the logarithmic regression model, you must rewrite the biologist's model with base e so that the inverse will involve a natural logarithm. This means that you want to find a constant c such that $e^c = 1.12$. Writing the exponential equation $e^c = 1.12$ in logarithmic form gives $c = \ln 1.12$, so $c = \boxed{}$ to the nearest thousandth.

Replacing 1.12 with $e^{\boxed{}}$ in the biologist's model gives $P = 62\left(e^{\boxed{}}\right)^t$, or $P = 62\,e^{\boxed{}\,t}$. Now find the inverse of this function.

Write the equation.	$P = 62e^{\boxed{}\,t}$
Divide both sides by 62.	$\frac{P}{62} = e^{\boxed{}\,t}$
Write in logarithmic form.	$\ln \frac{P}{62} = \boxed{}\,t$
Divide both sides by $\boxed{}$.	$\boxed{} \ln \frac{P}{62} = t$

So, the inverse of the exponential model is $t = \boxed{} \ln \frac{P}{62}$. To compare this model with the logarithmic regression model, use a graphing calculator to graph both $y = \boxed{} \ln \frac{x}{62}$ and $y = -35.6 + 8.66 \ln x$. You observe that the graphs [roughly coincide/significantly diverge], so the models are [basically equivalent/very different].

Reflect

3. **Discussion** In a later lesson, you will learn the quotient property of logarithms, which states that $\log_b \frac{m}{n} = \log_b m - \log_b n$ for any positive numbers m and n. Explain how you can use this property to compare the two models in Example 3.

Your Turn

4. Maria made a deposit in a bank account and left the money untouched for several years. The table lists her account balance at the end of each year.

Years Since the Deposit Was Made	Account Balance
0	\$1000.00
1	\$1020.00
2	\$1040.40
3	\$1061.21

a. Write an exponential model for the account balance as a function of time (in years since the deposit was made).

b. Find the inverse of the exponential model after rewriting it with a base of e. Describe what information the inverse gives.

c. Perform logarithmic regression on the data (using the account balance as the independent variable and time as the dependent variable). Compare this model with the inverse model from part b.

Elaborate

5. Which transformations of $f(x) = \log_b(x)$ change the function's end behavior (both as x increases without bound and as x decreases toward 0 from the right)? Which transformations change the location of the graph's x-intercept?

6. How are reference points helpful when graphing transformations of $f(x) = \log_b(x)$?

7. What are two ways to obtain a logarithmic model for a set of data?

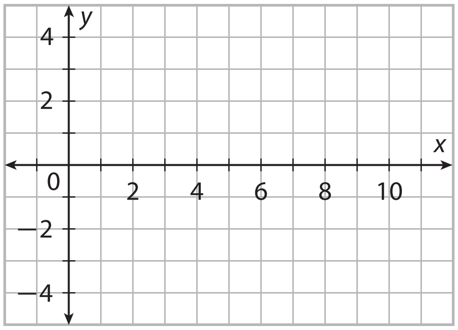
8. **Essential Question Check-In** Describe the transformations you must perform on the graph of $f(x) = \log_b(x)$ to obtain the graph of $g(x) = a\log_b(x - h) + k$.

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. Graph the logarithmic functions $f(x) = \log_2 x$, $f(x) = \log x$, and $f(x) = \ln x$ on the same coordinate plane. To distinguish the curves, label the point on each curve where the y-coordinate is 1.



2. Describe the attributes that the logarithmic functions $f(x) = \log_2 x$, $f(x) = \log x$, and $f(x) = \ln x$ have in common and the attributes that make them different. Attributes should include domain, range, end behavior, asymptotes, intercepts, intervals where the functions are positive and where they are negative, intervals where the functions are increasing and where they are decreasing, and the average rate of change on an interval.

3. For each of the six functions, describe how its graph is a transformation of the graph of $f(x) = \log_2 x$. Also identify what attributes of $f(x) = \log_2 x$ change as a result of the transformation. Attributes to consider are the domain, the range, the end behavior, the vertical asymptote, the x-intercept, the intervals where the function is positive and where it is negative, and whether the function increases or decreases throughout its domain.

a. $g(x) = \log_2 x - 5$

b. $g(x) = 4 \log_2 x$

c. $g(x) = \log_2 (x + 6)$

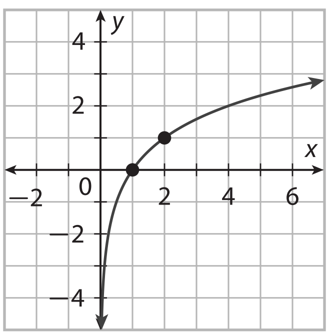
d. $g(x) = -\frac{3}{4} \log_2 x$

e. $g(x) = \log_2 x + 7$

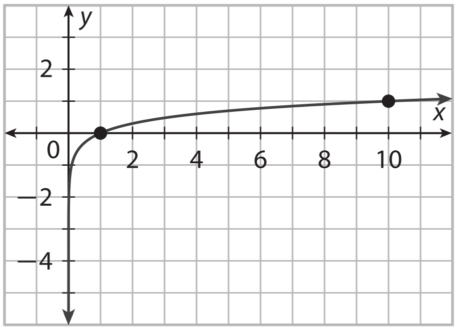
f. $g(x) = \log_2 (x - 8)$

Identify the transformations of the graph of $f(x) = \log_b x$ that produce the graph of the given function $g(x)$. Then graph $g(x)$ on the same coordinate plane as the graph of $f(x)$ by applying the transformations to the asymptote $x = 0$ and to the reference points $(1, 0)$ and $(b, 1)$. Also state the domain and range of $g(x)$ using set notation.

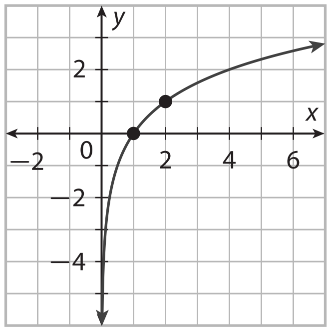
4. $g(x) = -4 \log_2 (x + 2) + 1$



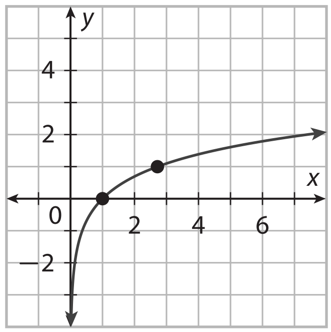
5. $g(x) = 3\log(x-1) - 1$



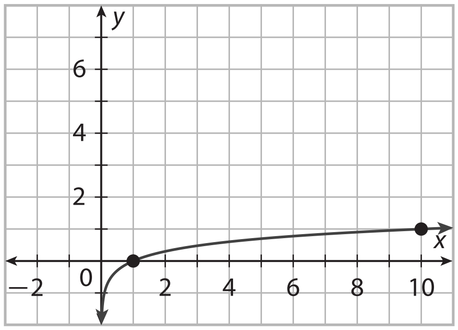
6. $f(x) = \frac{1}{2}\log_2(x-1) - 2$



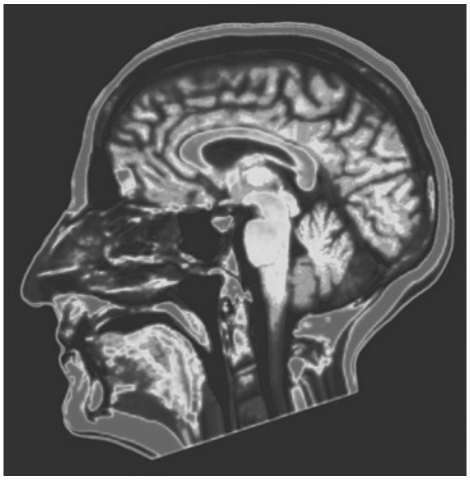
7. $g(x) = -4\ln(x-4) + 3$



8. $g(x) = -2\log(x+2) + 5$



9. The radioactive isotope fluorine-18 is used in medicine to produce images of internal organs and detect cancer. It decays to the stable element oxygen-18. The table gives the percent of fluorine-18 that remains in a sample over a period of time.



Time (hours)	Percent of Fluorine-18 Remaining
0	100
1	68.5
2	46.9
3	32.1

a. Write an exponential model for the percent of fluorine-18 remaining as a function of time (in hours).

b. Find the inverse of the exponential model after rewriting it with a base of *e*. Describe what information the inverse gives.

c. Perform logarithmic regression on the data (using the percent of fluorine-18 remaining as the independent variable and time as the dependent variable). Compare this model with the inverse model from part b.

10. During the period between 2001–2011, the average price of an ounce of gold doubled every 4 years. In 2001, the average price of gold was about \$270 per ounce.

Year	Average Price of an Ounce of Gold
2001	\$271.04
2002	\$309.73
2003	\$363.38
2004	\$409.72
2005	\$444.74
2006	\$603.46
2007	\$695.39
2008	\$871.96
2009	\$972.35
2010	\$1224.53
2011	\$1571.52

a. Write an exponential model for the average price of an ounce of gold as a function of time (in years since 2001).

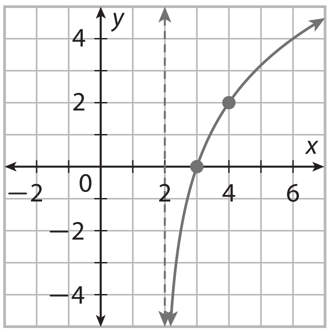
b. Find the inverse of the exponential model after rewriting it with a base of *e*. Describe what information the inverse gives.

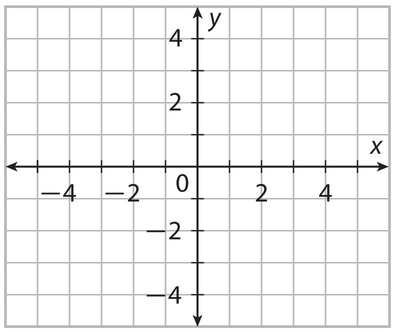
c. Perform logarithmic regression on the data in the table (using the average price of gold as the independent variable and time as the dependent variable). Compare this model with the inverse model from part b.

H.O.T. Focus on Higher Order Thinking

11. Multiple Representations For the function $g(x) = \log(x - h)$, what value of the parameter h will cause the function to pass through the point $(7, 1)$? Answer the question in two different ways: once by using the function's rule, and once by thinking in terms of the function's graph.

12. Explain the Error A student drew the graph of $g(x) = 2\log_{\frac{1}{2}}(x - 2)$ as shown. Explain the error that the student made, and draw the correct graph.





13. Construct Arguments Prove that $\log_{\frac{1}{b}} x = -\log_b x$ for any positive value of b not equal to 1. Begin the proof by setting $\log_{\frac{1}{b}} x$ equal to m and rewriting the equation in exponential form.

Lesson Performance Task

Given the following data about the heights of chair seats and table tops for children, make separate scatterplots of the ordered pairs (age of child, chair seat height) and the ordered pairs (age of child, table top height). Explain why a logarithmic model would be appropriate for each data set. Perform a logarithmic regression on each data set, and describe the transformations needed to obtain the graph of the model from the graph of the parent function $f(x) = \ln x$.

Age of Child (years)	Chair Seat Height (inches)	Table Top Height (inches)
1	5	12
1.5	6.5	14
2	8	16
3	10	18
5	12	20
7.5	14	22
11	16	25