Name ______________________ Class ____________ Date ________

14.1 Fitting Exponential Functions to Data

Essential Question: What are ways to model data using an exponential function of the form $f(x) = ab^x$?

Resource Locker

Explore Identifying Exponential Functions from Tables of Values

Notice for an exponential function $f(x) = ab^x$ that $f(x + 1) = ab^{x+1}$. By the product of powers property, $ab^{x+1} = a(b^x \cdot b^1) = ab^x \cdot b = f(x) \cdot b$. So, $f(x + 1) = f(x) \cdot b$. This means that increasing the value of x by 1 multiplies the value of $f(x)$ by b. In other words, for successive integer values of x, each value of $f(x)$ is b times the value before it, or, equivalently, the ratio between successive values of $f(x)$ is b. This gives you a test to apply to a given set of data to see whether it represents exponential growth or decay.

Each table gives function values for successive integer values of x. Find the ratio of successive values of $f(x)$ to determine whether each set of data can be modeled by an exponential function.

(A)

x	0	1	2	3	4
$f(x)$	1	4	16	64	256

$\frac{f(1)}{f(0)} = \square$; $\frac{f(2)}{f(1)} = \square$; $\frac{f(3)}{f(2)} = \square$; $\frac{f(4)}{f(3)} = \square$

The data are/are not exponential.

(B)

x	0	1	2	3	4
$f(x)$	1	7	13	19	25

$\frac{f(1)}{f(0)} = \square$; $\frac{f(2)}{f(1)} = \square$; $\frac{f(3)}{f(2)} = \square$; $\frac{f(4)}{f(3)} = \square$

The data are/are not exponential.

(C)

x	0	1	2	3	4
$f(x)$	1	4	13	28	49

$\frac{f(1)}{f(0)} = \square$; $\frac{f(2)}{f(1)} = \square$; $\frac{f(3)}{f(2)} = \square$; $\frac{f(4)}{f(3)} = \square$

The data are/are not exponential.

(D)

x	0	1	2	3	4
$f(x)$	1	0.25	0.0625	0.015625	0.00390625

$\frac{f(1)}{f(0)} = \square$; $\frac{f(2)}{f(1)} = \square$; $\frac{f(3)}{f(2)} = \square$; $\frac{f(4)}{f(3)} = \square$

The data [are/are not] exponential.

Reflect

1. In which step(s) does the table show exponential growth? Which show(s) exponential decay? What is the base of the growth or decay?

2. In which step are the data modeled by the exponential function $f(x) = 4^{-x}$?

3. What type of function model would be appropriate in each step not modeled by an exponential function? Explain your reasoning.

4. **Discussion** In the introduction to this Explore, you saw that the ratio between successive terms of $f(x) = ab^x$ is b. Find and simplify an expression for $f(x + c)$ where c is a constant. Then explain how this gives you a more general test to determine whether a set of data can be modeled by an exponential function.

Explain 1 Roughly Fitting an Exponential Function to Data

As the answer to the last Reflect question above indicates, if the ratios of successive values of the dependent variable in a data set for equally-spaced values of the independent variable are equal, an exponential function model fits. In the real world, sets of data rarely fit a model perfectly, but if the ratios are approximately equal, an exponential function can still be a good model.

Example 1

(A) **Population Statistics** The table gives the official population of the United States for the years 1790 to 1890.

Year	Total Population
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,860,702
1840	17,063,353
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	50,189,209
1890	62,979,766

Create an approximate exponential model for the data set. Then graph your function with a scatter plot of the data and assess its fit.

It appears that the ratio of the population in each decade to the population of the decade before it is pretty close to one and one third, so an exponential model should be reasonable.

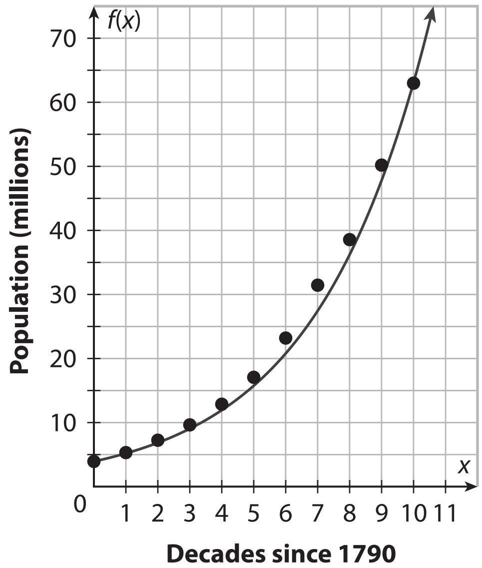
For a model of the form $f(x) = ab^x$, $f(0) = a$. So, if x is the number of decades after 1790, the value when $x = 0$ is a, the initial population in 1790, or 3,929,214.

One way to estimate the growth factor, b, is to find the population ratios from decade to decade and average them:

$$\frac{1.35 + 1.36 + 1.33 + 1.33 + 1.33 + 1.36 + 1.36 + 1.23 + 1.30 + 1.26}{10} \approx 1.32$$

An approximate model is $f(x) = 3.93(1.32)^x$, where $f(x)$ is in millions.

The graph is shown.



The graph looks like a good fit for the data. All of the points lie on, or close to, the curve.

Another way to estimate b is to choose a point other than $(0, a)$ from the scatter plot that appears would lie on, or very close to, the best-fitting exponential curve. Substitute the coordinates in the general formula and solve for b. For the plot shown, the point (8, 38.56) looks like a good choice.

$$38.56 = 3.93 \cdot b^8$$

$$(9.81)^{\frac{1}{8}} \approx \left(b^8\right)^{\frac{1}{8}}$$

$$1.33 \approx b$$

This value of b results in a model very similar to the previous model.

Ⓑ **Movies** The table shows the decline in weekly box office revenue from its peak for one of 2013's top-grossing summer movies.

Week	Revenue (in Millions of Dollars)
0	95.3
1	55.9
2	23.7
3	16.4
4	8.8
5	4.8
6	3.3
7	1.9
8	1.1
9	0.6

Create an approximate exponential model for the data set. Then graph your function with a scatter plot of the data and assess its fit.

Find the value of a in $f(x) = ab^x$.

When $x = 0$, $f(x) =$ ________. So, $a =$ ________.

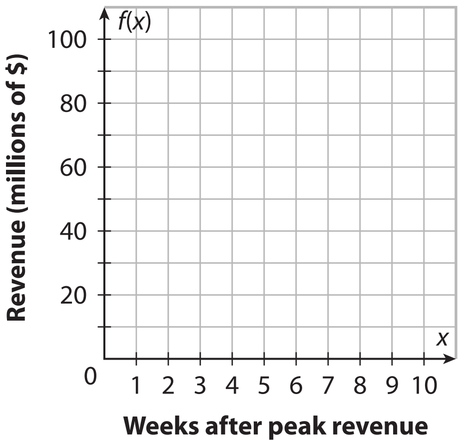
Find an estimate for b.

Approximate the revenue ratios from week to week and average them:

$$\frac{0.59 + 0.42 + 0.69 + 0.54 + \boxed{}}{\boxed{}} \approx \boxed{}$$

An approximate model is $f(x) = \boxed{}$.

The graph looks like a very good fit for the data. All of the points except one (2 weeks after peak revenue) lie on, or very close to, the curve.

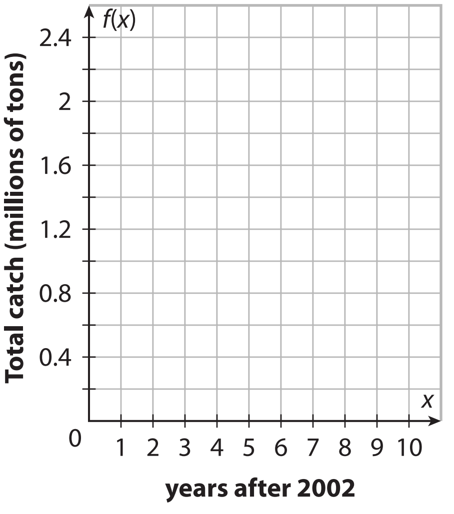


Your Turn

5. Fisheries The total catch in tons for Iceland's fisheries from 2002 to 2010 is shown in the table.

Year	Total Catch (Millions of Tons)
2002	2.145
2003	2.002
2004	1.750
2005	1.661
2006	1.345
2007	1.421
2008	1.307
2009	1.164
2010	1.063

Create an approximate exponential model for the data set. Then graph your function with a scatter plot of the data and assess its fit.



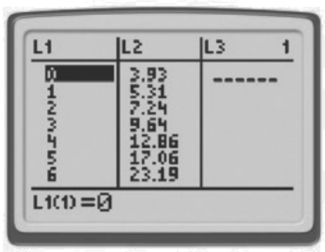
Explain 2 Fitting an Exponential Function to Data Using Technology

Previously you have used a graphing calculator to find a linear regression model of the form $y = ax + b$ to model data, and have also found quadratic regression models of the form $y = ax^2 + bx + c$. Similarly, you can use a graphing calculator to perform exponential regression to produce a model of the form $f(x) = ab^x$.

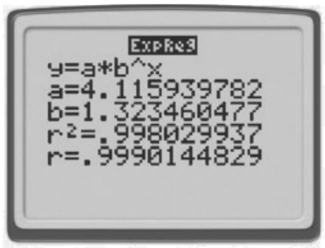
Example 2

Ⓐ **Population Statistics** Use the data from Example 1 Part A and a graphing calculator to find the exponential regression model for the data, and show the graph of the model with the scatter plot.

Using the STAT menu, enter the number of decades since 1790 in List1 and the population to the nearest tenth of a million in List2.

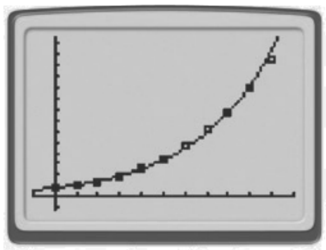


Using the STAT CALC menu, choose "ExpReg" and press ENTER until you see this screen:

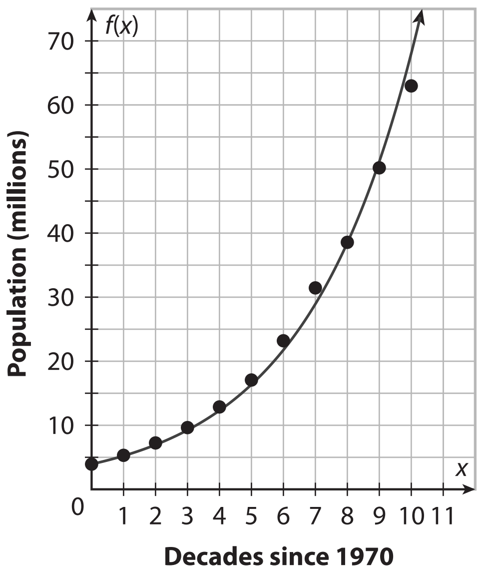


An approximate model is $f(x) = 4.116(1.323)^x$.

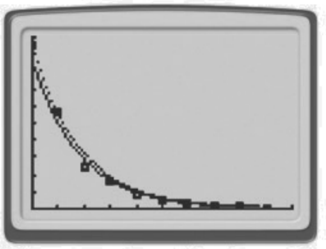
Making sure that STATPLOT is turned "On," enter the model into the Y = menu either directly or using the VARS menu and choosing "Statistics," "EQ," and "RegEQ." The graphs are shown using the ZoomStat window:



Plotted with the second graph from Example 1 (shown dotted), you can see that the graphs are nearly identical.



Ⓑ **Movies** Use the data from Example 1 Part B and a graphing calculator to find the exponential regression model for the data. Graph the regression model on the calculator, then graph the model from Example 1 on the same screen using a dashed curve. How do the graphs of the models compare? What can you say about the actual decline in revenue from one week after the peak to two weeks after the peak compared to what the regression model indicates?



Enter the data and perform exponential regression.

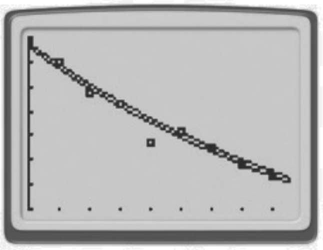
The model (using 3 digits of precision) is $f(x) =$ ______.

Reflect

6. **Discussion** The U.S. population in 2014 was close to 320 million people. What does the regression model in Part A predict for the population in 2014? What does this tell you about extrapolating far into the future using an exponential model? How does the graph of the scatter plot with the regression model support this conclusion? (Note: The decade-to-decade U.S. growth dropped below 30% to stay after 1880, and below 20% to stay after 1910. From 2000 to 2010, the rate was below 10%.)

Your Turn

7. **Fisheries** Use the data from YourTurn5 and a graphing calculator to find the exponential regression model for the data. Graph the regression model on the calculator, then graph the model from your answer to YourTurn5 on the same screen. How do the graphs of the models compare?



Explain 3 Solving a Real-World Problem Using a Fitted Exponential Function

In the real world, the purpose of finding a mathematical model is to help identify trends or patterns, and to use them to make generalizations, predictions, or decisions based on the data.

Example 3

(A) The Texas population increased from 20.85 million to 25.15 million from 2000 to 2010.

a. Assuming exponential growth during the period, write a model where $x = 0$ represents the year 2000 and $x = 1$ represents the year 2010. What was the growth rate over the decade?

b. Use the power of a power property of exponents to rewrite the model so that b is the yearly growth factor instead of the growth factor for the decade. What is the yearly growth rate for this model? Verify that the model gives the correct population for 2010.

c. The Texas population was about 26.45 million in 2013. How does this compare with the prediction made by the model?

d. Find the model's prediction for the Texas population in 2035. Do you think it is reasonable to use this model to guide decisions about future needs for water, energy generation, and transportation needs. Explain your reasoning.

a. For a model of the form $f(x) = ab^x$, a $= f(0)$, so $a = 20.85$. To find an estimate for b, substitute $\left(x, f(x)\right) = (1, 25.5)$ and solve for b.

$$f(x) = a \cdot b^x$$

$$25.15 = 20.85 \cdot b^1$$

$$\frac{25.15}{20.85} = b$$

$$1.206 \approx b$$

An approximate model is $f(x) = 20.85(1.206)^x$. The growth rate was about 20.6%.

b. Because there are 10 years in a decade, the 10th power of the new b must give 1.206, the growth factor for the decade. So, $b^{10} = 1.206$, or $b = 1.206^{\frac{1}{10}}$. Use the power of a power property:

$$f(x) = 20.85(1.206)^x = 20.85\left(1.206^{\frac{1}{10}}\right)^{10x} \approx 20.85(1.019)^x$$

Because x is decades after 2000, this is equivalent to $f(x) = 20.85(1.019)^x$ where x is years after 2000.

The model gives a 2010 population of $f(x) = 20.85(1.019)^{10} \approx 25.17$. This agrees with the actual population within a rounding error.

c. Substitute $x = 13$ into the model $f(x) = 20.85(1.019)^x$:

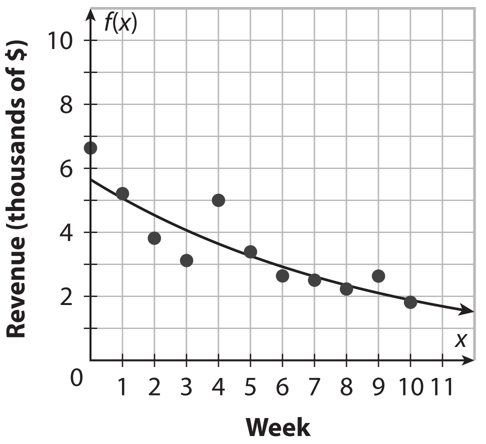
$$f(13) = 20.85(1.019)^{13} \approx 26.63$$

The prediction is just a little bit higher than the actual population.

d. For 2035, $x = 35$: $f(35) = 20.85(1.019)^{35} \approx 40.3$. The model predicts a Texas population of about 40 million in 2035. Possible answer: Because it is very difficult to maintain a high growth rate with an already very large population, and with overall population growth slowing, it seems unreasonable that the population would increase from 25 to 40 million so quickly. But because using the model to project even to 2020 gives a population of over 30 million, it seems reasonable to make plans for the population to grow by several million people over a relatively short period.

(B) The average revenue per theater for the movie in Part B of the previous Examples is shown in the graph. (Note that for this graph, Week 0 corresponds to Week 2 of the graphs from the previous Examples.) The regression model is $y = 5.65(0.896)^x$.

a. From Week 3 to Week 4, there is a jump of over 60% in the average weekly revenue per theater, but the total revenue for the movie for the corresponding week fell by over 30%. What must have occurred for this to be true?



b. A new theater complex manager showing a similar summer movie in a single theater worries about quickly dropping revenue the first few weeks, and wants to stop showing the movie. Suppose you are advising the manager. Knowing that the model shown reflects the long-term trend well for such movies, what advice would you give the manager?

Reflect

8. **Discussion** Consider the situation in Example 3B about deciding when to stop showing the movie. How does an understanding of what other theater managers might do affect your decision?

Your Turn

9. Graph the regression model for the catch in Icelandic fisheries, $f(x) = 2.119(0.9174^x)$, to find when the model predicts the total catches to drop below 0.5 million tons (remember that $x = 0$ corresponds to 2002). Should the model be used to project actual catch into the future? Why or why not? What are some considerations that the model raises about the fishery?

Elaborate

10. How can you tell whether a given set of data can reasonably be modeled using an exponential function?

11. What are some ways that an exponential growth or decay model can be used to guide decisions, preparations, or judgments about the future?

12. **Essential Question Check-In** What are some ways to find an approximate exponential model for a set of data without using a graphing calculator?

Evaluate: Homework and Practice

• Online Homework
• Hints and Help
• Extra Practice

Determine whether each set of data can be modeled by an exponential function. If it can, tell whether it represents exponential growth or exponential decay. If it can't, tell whether a linear or quadratic model is instead appropriate.

1.

x	0	1	2	3	4
$f(x)$	2	6	18	54	162

2.

x	1	2	3	4	5	6
$f(x)$	1	2	3	5	8	13

3.

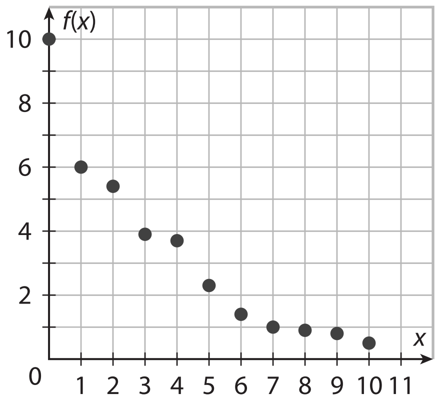
x	0	1	2	3	4
$f(x)$	2	8	18	32	50

4.

x	5	10	15	20	25
$f(x)$	76.2	66.2	59.1	50.9	44.6

Three students, Anja, Ben, and Celia, are asked to find an approximate exponential model for the data shown. Use the data and scatter plot for Exercises 5–7.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	10	6.0	5.4	3.9	3.7	2.3	1.4	1.0	0.9	0.8	0.5



5. To find an approximate exponential model, Anja uses the first data point to find a, and then estimates b by finding the ratio of the first two function values. What is her model?

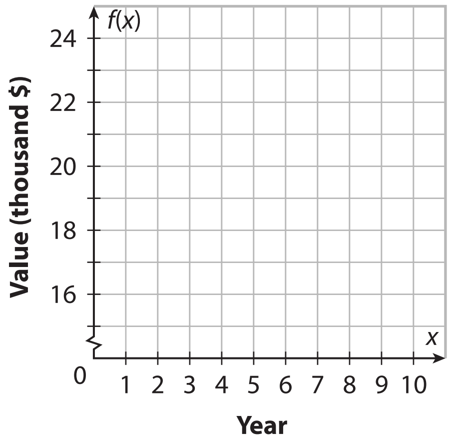
6. To find his model, Ben uses the first and last data points. What is his model?

7. Celia thinks that because the drop between the first two points is so large, the best model might actually have a y-intercept a little below 10. She uses $(0, 9.5)$ to estimate a in her model. To estimate b, she finds the average of the ratios of successive data values. What is her model? (Use two digits of precision for all quantities.)

8. **Classic Cars** The data give the estimated value in dollars of a model of classic car over several years.

15,300	16,100	17,300	18,400	19,600	20,700	22,000

a. Find an approximate exponential model for the car's value by averaging the successive ratios of the value. Then make a scatter plot of the data, graph your model with the scatter plot, and assess its fit to the data.

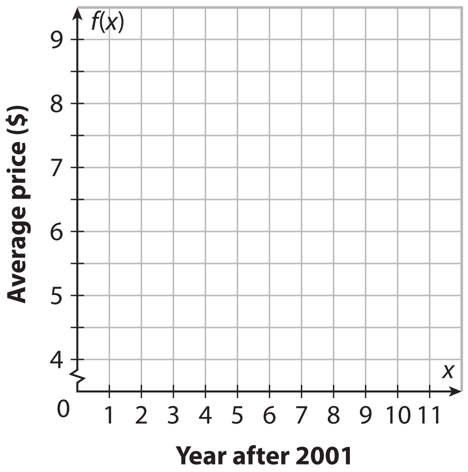


b. In the last year of the data, a car enthusiast spends $15,100 on a car of the given model that is in need of some work. The owner then spends $8300 restoring it. Use your model to create a table of values with a graphing calculator. How long does the function model predict the owner should keep the car before it can be sold for a profit of at least $5000?

9. **Movies** The table shows the average price of a movie ticket in the United States from 2001 to 2010.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Price ($)	5.66	5.81	6.03	6.21	6.41	6.55	6.88	7.18	7.50	7.89

a. Make a scatter plot of the data. Then use the first point and another point on the plot to find an approximate exponential model for the average ticket price. Then graph the model with your scatter plot and assess its fit to the data.



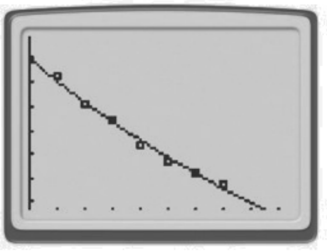
b. Use a graphing calculator to find a regression model for the data, and graph the model with the scatter plot. How does this model compare to your previous model?

c. What does the regression model predict for the average cost in 2014? How does this compare with the actual 2014 cost of about $8.35? A theater owner uses the model in 2010 to project income for 2014 assuming average sales of 490 tickets per day at the predicted price. If the actual price is instead $8.35, did the owner make a good business decision? Explain.

10. Pharmaceuticals A new medication is being studied to see how quickly it is metabolized in the body. The table shows how much of an initial dose of 15 milligrams remains in the bloodstream after different intervals of time.

Hours Since Administration	Amount Remaining (mg)
0	15
1	14.3
2	13.1
3	12.4
4	11.4
5	10.7
6	10.2
7	9.8

a. Use a graphing calculator to find a regression model. Use the calculator to graph the model with the scatter plot. How much of the drug is eliminated each hour?



b. The half-life of a drug is how long it takes for half of the drug to be broken down or eliminated from the bloodstream. Using the Table function, what is the half-life of the drug to the nearest hour?

c. Doctors want to maintain at least 7 mg of the medication in the bloodstream for maximum therapeutic effect, but do not want the amount much higher for a long period. This level is reached after 12 hours. A student suggests that this means that a 15 mg dose should be given every 12 hours. Explain whether you agree with the student. (*Hint*: Given the medicine's decay factor, how much will be in the bloodstream after the first few doses?)

11. **Housing** The average selling price of a unit in a high-rise condominium complex over 5 consecutive years was approximately \$184,300; \$195,600; \$204,500; \$215,300; \$228,200.

a. Find an exponential regression model where x represents years after the initial year and $f(x)$ is in thousands of dollars.

b. A couple wants to buy a unit in the complex. First, they want to save 20% of the selling price for a down payment. What is the model that represents 20% of the average selling price for a condominium?

c. At the time that the average selling price is \$228,200 (or when $x = 4$), the couple has \$20,000 saved toward a down payment. They are living with family, and saving \$1000 per month. Graph the model from Part b and a function that represents the couple's total savings on the same calculator screen. How much longer does the model predict it will take them to save enough money?

12. **Business growth** The growth in membership in thousands of a rapidly-growing Internet site over its first few years is modeled by $f(x) = 60(3.61)^x$ where x is in years and $x = 0$ represents the first anniversary of the site. Rewrite the model so that the growth factor represents weeks instead of years. What is the weekly growth factor? What does this model predict for the membership 20 weeks after the anniversary?

13. Which data set can be modeled by an exponential function $f(x) = ab^x$?

a. $(0, 0.1), (1, 0.5), (2, 2.5), (3, 12.5)$

b. $(0, 0.1), (1, 0.2), (2, 0.3), (3, 0.4)$

c. $(0, 1), (1, 2), (2, 4), (4, 8)$

d. $(0, 0.8), (1, 0.4), (2, 0.10), (3, 0.0125)$

H.O.T. Focus on Higher Order Thinking

14. **Error analysis** From the data $(2, 72.2), (3, 18.0), (4, 4.4), (5, 1.1), (6, 0.27)$, a student sees that the ratio of successive values of $f(x)$ is very close to 0.25, so that an exponential model is appropriate. From the first term, the student obtains $a = 72.2$, and writes the model $f(x) = 72.2(0.25)^x$. The student graphs the model with the data and observes that it does not fit the data well. What did the student do wrong? Correct the student's model.

15. **Critical thinking** For the data $(0, 5), (1, 4), (2, 3.5), (3, 3.25), (4, 3.125), (5, 3.0625)$, the ratio of consecutive y-values is not constant, so you cannot write an exponential model $f(x) = ab^x$. But the difference in the values from term to term, 1, 0.5, 0.25, 0.125, 0.0625, shows exponential decay with a decay factor of 0.5. How can you use this fact to write a model of the data that contains an exponential expression of the form ab^x?

16. **Challenge** Suppose that you have two data points (x_1, y_1) and (x_2, y_2) that you know are fitted by an exponential model $f(x) = ab^x$. Can you always find an equation for the model? Explain.

Lesson Performance Task

According to data from the U.S. Department of Agriculture, the number of farms in the United States has been decreasing over the past several decades. During this time, however, the average size of each farm has increased.

Farms in the United States	
Year	**Farms (Millions)**
1940	6.35
1950	5.65
1960	3.96
1970	2.95
1980	2.44
1990	2.15
2000	2.17

a. The average size in acres of a U.S. farm from 1940 to 1980 can be modeled by the function $A(t) = 174e^{0.022t}$ where t is the number of years since 1940. What was the average farm size in 1940? In 1980?

b. The table shows the number of farms in the United States from 1940 to 2000. Find an exponential model for the data using a calculator.

c. If you were to determine the exponential model without a calculator, would the value for a be the same as the value from the calculator? Explain your answer.

d. Based on the data in the table, predict the number of farms in the United States in 2014.

e. Using a graphing calculator, determine how many years it takes for the number of farms to decrease by 50%.

f. Using a graphing calculator, determine when the number of farms in the United States will fall below 1 million.

g. Does an exponential model seem appropriate for all of the data listed in the table? Why or why not?