Name ______________________ Class ____________ Date ________

12.3 Geometric Series

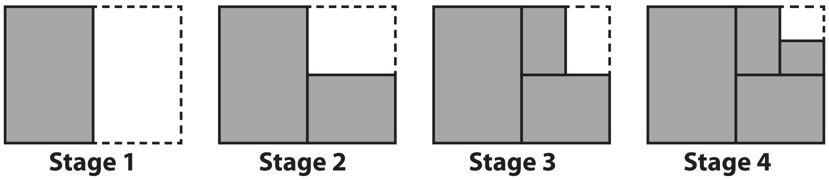
Essential Question: How do you find the sum of a finite geometric series?

Resource Locker

Explore 1 Investigating a Geometric Series

A **series** is the expression formed by adding the terms of a sequence. If the sequence is geometric and has a finite number of terms, it is called a **finite geometric series**. In this Explore, you will generate several related finite geometric series and find a formula for calculating the sum of each series.

(A) Start with a rectangular sheet of paper and assume the sheet has an area of 1 square unit. Cut the sheet in half and lay down one of the half-pieces. Then cut the remaining piece in half, and lay down one of the quarter-pieces as if rebuilding the original sheet of paper. Continue the process: At each stage, cut the remaining piece in half, and lay down one of the two pieces as if rebuilding the original sheet of paper.



(B) Complete the table by expressing the total area of the paper that has been laid down in two ways:

- as the sum of the areas of the pieces that have been laid down, and
- as the difference between 1 and the area of the remaining piece.

Stage	Sum of the areas of the pieces that have been laid down	Difference of 1 and the area of the remaining piece
1	$\frac{1}{2}$	$1 - \frac{1}{2} = \square$
2	$\frac{1}{2} + \square = \square$	$1 - \square = \square$
3	$\frac{1}{2} + \square + \square = \square$	$1 - \square = \square$
4	$\frac{1}{2} + \square + \square + \square = \square$	$1 - \square = \square$

Reflect

1. Write the sequence formed by the areas of the individual pieces that are laid down. What type of sequence is it?

2. In the table from Step B, you wrote four related finite geometric series: $\frac{1}{2}$, $\frac{1}{2}+\frac{1}{4}$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, and $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$. One way you found the sum of each series was simply to add up the terms. Describe another way you found the sum of each series.

3. If the process of cutting the remaining piece of paper and laying down one of the two pieces is continued, you obtain the finite geometric series $\frac{1}{2}+\frac{1}{4}+\ldots+\left(\frac{1}{2}\right)^n$ at the nth stage. Use your answer to the previous question to find the sum of this series.

Explore 2 Deriving a Formula for the Sum of a Finite Geometric Series

To find a general formula for the sum of a finite geometric series with n terms, begin by writing the series as $S(n) = a + ar + ar^2 + \ldots + ar^{n-1}$.

(A) Find an expression for $rS(n)$.

$$rS(n) = ar + ar^2 + ar^{\square} \ldots + ar^{\square}$$

(B) Find an expression for $S(n) - rS(n)$ by aligning like terms and subtracting.

$$S(n) = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rS(n) = ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

$$S(n) - rS(n) = a + \square + \square + \ldots + \square - ar^n$$

(C) Simplify the expression for $S(n) - rS(n)$.

$$S(n) - rS(n) = \square$$

(D) Factor the left and right sides of the equation in Step C.

$$S(n)\square = \square(1 - r^{\square})$$

(E) Divide both sides of the equation in Step D by $1 - r$.

$$S(n) = a\left(\frac{\square - \square}{1 - r}\right)$$

Reflect

4. Check to see if the formula in Step E gives the same result as the answer you wrote for Reflect 3.

5. What restrictions are there on the values of r that can be used in the formula for the sum of a finite geometric series? Explain.

Explain 1 Finding the Sum of a Finite Geometric Series

The formula $S(n) = a\left(\frac{1-r^n}{1-r}\right)$ for the sum of a geometric series requires knowing the values of a, r, and n.

Recall that you learned how to find a and r for a geometric sequence, and the technique is no different for a series: a is the value of the first term, and r is the ratio of any two successive terms. To find n, you can simply count the terms if they are all listed. For instance, for the finite geometric series $3 + 6 + 12 + 24 + 48$, you can see that $a = 3$, $r = 2$, and $n = 5$.

If some of the terms of a finite geometric series have been replaced by an ellipsis, as in $2 + 6 + 18 + \ldots + 1458$, you obviously can't count the terms. One way to deal with this situation is to generate the missing terms by using the common ratio, which in this case is 3. The next term after 18 is $3(18) = 54$, and repeatedly multiplying by 3 to generate successive terms gives $2 + 6 + 18 + 54 + 162 + 486 + 1458$, so the series has 7 terms.

Another way to find the number of terms in $2 + 6 + 18 + \ldots + 1458$ is to recognize that the nth term in a geometric series is ar^{n-1}. For the series $2 + 6 + 18 + \ldots + 1458$ whose nth term is $2(3)^{n-1}$, find n as follows:

$2(3)^{n-1} = 1458$	Set the nth term equal to the last term.
$(3)^{n-1} = 729$	Divide both as power of 3
$(3)^{n-1} = 3^6$	Write 729 as a power of 3
$n - 1 = 6$	When the bases are the same, you can equate the exponents.
$n = 7$	Add 1 to both sides

Find the sum of the finite geometric series.

A $5 + 15 + 45 + 135 + 405 + 1215$

Step 1 Find the values of a, r, and n.

The first term in the series is a.	$a = 5$
Find the common ratio r by dividing two successive terms.	$r = \frac{15}{5} = 3$
Count the terms to find n.	$n = 6$

Step 2 Use the formula $S(n) = a\left(\frac{1 - r^n}{1 - r}\right)$.

Substitute the values of a, r, and n. $\quad S(6) = 5\left(\frac{1 - 3^6}{1 - 3}\right)$

Evaluate the power in the numerator. $\quad = 5\left(\frac{1 - 729}{1 - 3}\right)$

Simplify the numerator and denominator. $\quad = 5\left(\frac{-728}{-2}\right)$

Simplify the fraction. $\quad = 5(364)$

Multiply. $\quad = 1820$

B $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{512}$

Step 1 Find the values of a, r, and n.

The first term in the series is a. $\quad a = \boxed{}$

Find the common ratio by dividing two successive terms. $\quad r = \frac{\frac{1}{8}}{\boxed{}} = \boxed{}$

Set the ***n***th term, $\frac{1}{4}\left(\frac{1}{2}\right)^{n-1}$, equal to the last term to find n. $\quad \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} = \boxed{}$

Multiply both sides by $\boxed{}$. $\quad \left(\frac{1}{2}\right)^{n-1} = \frac{1}{128}$

Write $\frac{1}{128}$ as a power of $\frac{1}{2}$. $\quad \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{\boxed{}}$

Equate the exponents. $\quad n - 1 = \boxed{}$

Add 1 to both sides. $\quad n = \boxed{}$

Step 2 Use the formula $S(n) = a\left(\frac{1-r^n}{1 - r}\right)$.

Substitute the values of a, r, and n. $\quad S(8) = \boxed{}\left(\frac{1 - \left(\boxed{}\right)^{\boxed{}}}{1 - \boxed{}}\right)$

Evaluate the power in the numerator. $\quad = \frac{1}{4}\left(\frac{1 - \boxed{}}{1 - \frac{1}{2}}\right)$

Simplify the numerator and denominator. $\quad = \frac{1}{4}\left(\frac{\frac{\boxed{}}{256}}{\frac{1}{2}}\right)$

Simplify the fraction. $\quad = \frac{1}{4}\left(\boxed{}\right)$

Multiply. $\quad = \boxed{}$

Your Turn

Find the sum of the finite geometric series.

6. $1 - 2 + 4 - 8 + 16 - 32$

7. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} \dots - \frac{1}{256}$

Explain 2 Solving a Real-World Problem Involving a Finite Geometric Series

Some financial problems can be modeled by a geometric series. For instance, an *annuity* involves equal payments made at regular intervals for a fixed amount of time. Because money can be invested and earn interest, comparing the value of money today to the value of money in the future requires accounting for the effect of interest. The *future value* of an annuity is how much the annuity payments will be worth at some point in the future. The *present value* of an annuity is how much the annuity payments are worth in the present.

Although an interest rate is typically expressed as an annual rate, it can be converted to a rate for other periods of time. For instance, an annual interest rate of $r\%$ results in a monthly interest rate of $\frac{r}{12}\%$. In general, if interest is earned n times per year, an annual interest rate of $r\%$ is divided by n.

Example 2

(A) Niobe is saving for a down payment on a new car, which she intends to buy a year from now. At the end of each month, she deposits $200 from her paycheck into a dedicated savings account, which earns 3% annual interest that is applied to the account balance each month. After making 12 deposits, how much money will Niobe have in her savings account?

Niobe is interested in the future value of her annuity (savings plan). A 3% annual interest rate corresponds to a $\frac{3}{12}\% = 0.25\%$ monthly interest rate.

First, calculate the sequence of end-of-month account balances. Recognize the recursive nature of the calculations:

- The end-of-month balance for month 1 is $200 because the first deposit of $200 is made at the end of the month, but the deposit doesn't earn any interest that month.
- The end-of-month balance for any other month is the sum of the previous month's end-of-month balance, the interest earned on the previous month's end-of-month balance, and the next deposit.

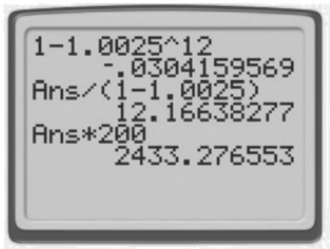
So, if $B(m)$ represents the account balance for month m, then a recursive rule for the account balances is $B(1) = 200$ and $B(m) = B(m-1) + B(m-1) \cdot 0.0025 + 200$. Notice that you can rewrite the equation $B(m) = B(m-1) + B(m-1) \cdot 0.0025 + 200$ as $B(m) = B(m-1) \cdot 1.0025 + 200$ by using the Distributive Property.

Month	End-of-month balance of account
1	200
2	$200 \cdot 1.0025 + 200$
3	$[200(1.0025) + 200] \cdot 1.0025 + 200 = 200(1.0025)^2 + 200(1.0025) + 200$
4	$[200(1.0025)^2 + 200(1.0025) + 200] \cdot 1.0025 + 200 = 200(1.0025)^3$ $+ 200\,(1.0025)^2 + 200(1.0025) + 200$
$\vdots$	$\vdots$
12	$[200(1.0025)^{10} + \cdots + 200] \cdot 1.0025 + 200 = 200(1.0025)^{11} + \cdots + 200(1.0025) + 200$

Next, find the sum of the finite geometric series that represents the end-of-month balance after 12 deposits. You may find it helpful to use the commutative property to rewrite $200(1.0025)^{11} + \cdots + 200(1.0025) + 200$ as $200 + 200(1.0025) + \cdots + 200(1.0025)^{11}$ so that it's easier to see that the initial term a is 200 and the common ratio r is 1.0025. Also, you know from the recursive process that this series has 12 terms. Apply the formula for the sum of a finite geometric series in order to obtain the final balance of the account.

$$S(12) = 200\left(\frac{1 - 1.0025^{12}}{1 - 1.0025}\right)$$

To evaluate the expression for the sum, use a calculator. You may find it helpful to enter the expression in parts and rely upon the calculator's Answer feature to accumulate the results. (You should avoid rounding intermediate calculations, because the round-off errors will compound and give an inaccurate answer.)



So, Niobe will have \$2433.28 in her account after she makes 12 deposits.

B Niobe decides to postpone buying a new car because she wants to get a new smart phone instead. She can pay the phone's full price of \$580 up front, or she can agree to pay an extra \$25 per month on her phone bill over the course of a two-year contract for phone service.

What is the present cost to Niobe if she agrees to pay \$25 per month for two years, assuming that she could put the money for the payments in a savings account that earns 3% annual interest and make \$25 monthly withdrawals for two years?

As in Part A, a 3% annual interest rate becomes a 0.25% monthly interest rate. If Niobe puts an amount M_1 in the savings account and lets it earn interest for 1 month, then she will have $M_1 + 0.0025M_1$, or $1.0025M_1$, available to make her first phone payment. Since she wants $1.0025M_1$ to equal \$25, M_1 must equal $\frac{\$25}{1.0025} \approx \$24.94$. This means that the present cost of her first phone payment is \$24.94, because that amount of money will be worth \$25 in 1 month after earning \$0.06 in interest.

Similarly, if Niobe puts an additional amount M_2 in the savings account and lets it earn interest for 2 months, then she will have $1.0025M_2$ after 1 month and $1.0025(1.0025M_2)$, or $(1.0025)^{\square}M_2$, after 2 months. Since she wants $(1.0025)^{\square}M_2$ to equal \$25, M_2 must equal $\dfrac{\$25}{(1.0025)^{\square}} \approx \$\square$.

This means that the present cost of her second phone payment is \$$\square$. It also means that she must deposit a total of $M_1 + M_2 = \$24.94 + \$\square = \$\square$ in the savings account in order to have enough money for her first two phone payments.

Generalize these results to complete the following table.

Number of Payments	Present Cost of Payments
1	$\dfrac{25}{1.0025}$
2	$\dfrac{25}{1.0025} + \dfrac{25}{(1.0025)^2}$
3	$\dfrac{25}{1.0025} + \dfrac{25}{(1.0025)^2} + \dfrac{25}{(1.0025)^{\square}}$
⋮	⋮
24	$\dfrac{25}{1.0025} + \dfrac{25}{(1.0025)^2} + \cdots + \dfrac{25}{(1.0025)^{\square}}$

Find the sum of the finite geometric series that represents the present cost of 24 payments.

$$a = \frac{25}{1.0025}$$

$$r = \square$$

$$n = \square$$

$$S(24) = \square\left(\frac{1 - \square^{\square}}{1 - \square}\right)$$

$$\approx 581.65$$

Although Niobe will end up making total payments of $\$25 \cdot 24 = \$\square$, the present cost of the payments is \$581.65, which is only slightly more than the up-front price of the phone.

YourTurn

8. A lottery winner is given the choice of collecting $1,000,000 immediately or collecting payments of $6000 per month for the next 20 years. Assuming the lottery money can be invested in an account with an annual interest rate of 6% that is applied monthly, find the present value of the lottery's delayed-payout plan in order to compare it with the lump-sum plan and decide which plan is better.

Elaborate

9. An alternative way of writing the formula for the sum of a finite geometric series is $S(n) = \dfrac{a - r \cdot ar^{n-1}}{1 - r}$. Describe in words how to find the sum of a finite geometric series using this formula.

10. Describe how to find the number of terms in a finite geometric series when some of the terms have been replaced by an ellipsis.

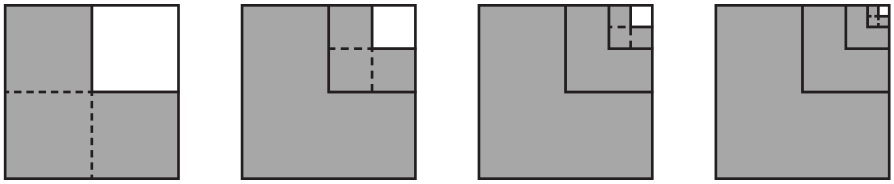
11. **Discussion** When analyzing an annuity, why is it important to determine the annuity's present value or future value?

12. **Essential Question Check-In** What is the formula for the sum of the finite geometric series $a + ar + ar^2 + \cdots + ar^{n-1}$?

Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

1. Suppose you start with a square piece of paper that you divide into four quarters, cutting out an L-shaped piece using three of the quarters and laying it down to create the first term of a geometric series. You then use the remaining quarter to repeat the process three more times, as shown.



Stage 1 **Stage 2** **Stage 3** **Stage 4**

a. Complete the table.

Stage	Sum of the areas of the pieces that have been laid down	Difference of 1 and the area of the remaining piece
1	$\frac{3}{4}$	$1 - \frac{1}{4} = \frac{3}{4}$
2		
3		
4		

b. Generalize the results in the table: At stage n, the second column gives you the finite geometric series $\frac{3}{4} + \frac{3}{16} + \cdots + 3\left(\frac{1}{4}\right)^n$. The third column gives you a way to find the sum of this series. What formula does the third column give you?

c. Show that the general formula for the sum of a finite geometric series agrees with the specific formula from part b.

2. In a later lesson you will learn how to use polynomial division to show that $\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$ for any integer n greater 0. Use this identity as an alternative method of deriving the formula for the sum of a finite geometric series with n terms. That is, given the series $a + ar + ar^2 + \cdots + ar^{n-1}$, show that its sum is $a\left(\frac{1 - r^n}{1 - r}\right)$.

Find the sum of the finite geometric series.

3. $-3 + 6 - 12 + 24 - 48 + 96 - 192 + 384$

4. $6 - 4 + \frac{8}{3} - \frac{16}{9} + \frac{32}{17}$

Determine how many terms the geometric series has, and then find the sum of the series.

5. $-12 - 4 - \frac{4}{3} - \cdots - \frac{4}{243}$

6. $0.3 + 0.03 + 0.003 + \cdots + 0.000003$

7. $6 + 30 + 150 + \cdots + 468{,}750$

8. $-3 + 9 - 27 + \cdots - 177{,}147$

Write the finite geometric series from its given description, and then find its sum.

9. A geometric series that starts with 2, ends with -6250, and has a common ratio of -5

10. A geometric series with 5 terms that begins with 1 and has a common ratio of $\frac{1}{3}$.

11. A geometric series with 7 terms that begins with 1000 and successively decreases by 20%

12. A geometric series where the first term is -12, the last term is -972, and each term after the first is triple the previous term

13. **Chess** The first international chess tournament was held in London in 1851. This single-elimination tournament (in which paired competitors played matches and only the winner of a match continued to the next round) began with 16 competitors. How many matches were played?

14. A ball is dropped from an initial height and allowed to bounce repeatedly. On the first bounce (one up-and-down motion), the ball reaches a height of 32 inches. On each successive bounce, the ball reaches 75% of its previous height. What is the total vertical distance that the ball travels in 10 bounces? (Do not include the initial height from which the ball is dropped.)

15. **Medicine** During a flu outbreak, health officials record 16 cases the first week, 56 new cases the second week, and 196 new cases the third week.

a. Assuming the pattern of new cases continues to follow a geometric sequence, what total number of new cases will have been recorded by the fifth week?

b. How many weeks will it take for the total number of recorded cases to exceed 40,000?

16. **Finance** A person deposits $5000 into an investment account at the end of each year for 10 years. The account earns 4% interest annually. What is the future value of the annuity after the 10th deposit?

17. Business A business wants to buy a parcel of land in order to expand its operations. The owner of the land offers two purchase options: Buy the land today for \$100,000, or buy the land in five equal payments of \$22,000 where the payments are due a year apart and the first payment is due immediately. The chief financial officer for the business determines that money set aside for the purchase of the land can be invested and earn 5.4% interest annually. Which purchase option is the better deal for the business? Explain.

18. Match each finite geometric series on the left with its sum on the right.

A. $2 + 6 + 18 + \cdots + 1458$ ________ 1094

B. $2 - 6 + 18 - \cdots + 1458$ ________ -2186

C. $-2 + 6 - 18 + \cdots - 1458$ ________ 2186

D. $-2 - 6 - 8 - \cdots - 1458$ ________ -1094

H.O.T. Focus on Higher Order Thinking

19. Represent Real-World Problems The formula for the future value FV of an annuity consisting of n regular payments of p dollars at an interest rate of i (expressed as a decimal) is $FV = p\left(\frac{(1 + i)^n - 1}{i}\right)$, which is valid for any payment rate (such as monthly or annually) as long as the interest rate has the same time unit. The formula assumes that the future value is calculated when the last payment is made. Show how to derive this formula.

20. Represent Real-World Problems The formula for the present value PV of an annuity consisting of n regular payments of p dollars at an interest rate of i (expressed as a decimal) is $PV = p\left(\frac{1-(1+i)^{-n}}{i}\right)$, which is valid for any payment rate (such as monthly or annually) as long as the interest rate has the same time unit. The formula assumes that the present value is calculated one time unit before the first payment is made. Show how to derive this formula.

21. Draw Conclusions Consider whether it's possible for the infinite geometric series $a + ar + ar^2 + \cdots$ to have a finite sum. Since the formula $S(n) = a\left(\frac{1-r^n}{1-r}\right)$ gives the sum of the first n terms of the series, a reasonable approach to finding the sum of all terms in the series is to determine what happens to $S(n)$ as n increases without bound. Use this approach on each of the following series and draw a conclusion.

a. $1 + 2 + 4 + 8 + \cdots$

b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Lesson Performance Task

You've finally purchased your dream home after saving for a long time. You've made a nice down payment, and your mortgage loan is \$150,000. It is a 30-year loan with an annual interest rate of 4.5%, which is calculated monthly. Find a formula for calculating monthly mortgage payments. Then find the monthly payment needed to pay off your mortgage loan.

Let P be the principal, r be the monthly interest rate expressed as a decimal, and m be the monthly payment.