Name ______________________ Class ______________ Date ________

12.2 Geometric Sequences

Essential Question: How can you define a geometric sequence algebraically?

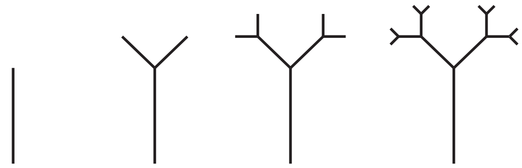
Resource Locker

Explore Investigating Geometric Sequences

As a tree grows, limbs branch off of the trunk, then smaller limbs branch off these limbs and each branch splits off into smaller and smaller copies of itself the same way throughout the entire tree. A mathematical object called a *fractal tree* resembles this growth.

Start by drawing a vertical line at the bottom of a piece of paper. This is Stage 0 of the fractal tree and is considered to be one 'branch'. The length of this branch defines 1 unit.

For Stage 1, draw 2 branches off of the top of the first branch. For this fractal tree, each smaller branch is $\frac{1}{2}$ the length of the previous branch and is at a 45-degree angle from the direction of the parent branch. The first four iterations, Stages 0-3, are shown.



Ⓐ In Stage 2, there are 2 branches drawn on the end of each of the 2 branches drawn in Stage 1. There are ☐ new branches in Stage 2. Each one of these branches will be $\frac{1}{2}$ the length of its predecessors or ☐ unit in length.

Ⓑ For Stage 3, there are 8 new branches in total. To draw Stage 4, a total of ☐ branches must be drawn and to draw Stage 5, a total of ☐ branches must be drawn. Thus, each stage adds ☐ times as many branches as the previous stage did.

Ⓒ Complete the table.

Stage	New Branches	Pattern	New Branches as a Power
Stage 0	1	1	2^0
Stage 1	2	$2 \cdot 1$	2^1
Stage 2	4	$2 \cdot 2$	2^2
Stage 3	8	$2 \cdot$	
Stage 4	16	$2 \cdot$	
Stage 5	32	$2 \cdot$	
Stage 6	64	$2 \cdot$	

D. The procedure for each stage after Stage 0 is to draw ______ branches on ______ branch added in the previous step.

E. Using the description above, write an equation for the number of new branches in a stage given the previous stage. Represent stage s as N_s; Stage 3 will be N_3.

$N_4 = \square \cdot \square$ $\qquad$ $N_6 = \square \cdot \square$

$N_5 = \square \cdot \square$ $\qquad$ $N_s = \square \cdot \square$

F. Rewrite the rule for Stage s as a function $N(s)$ that has a stage number as an input and the number of new branches in the stage as an output.

$N(s) = \square$

G. Recall that the domain of a function is the set of all numbers for which the function is defined. $N(s)$ is a function of s and s is the stage number. Since the stage number refers to the ______________ the tree has branched, it has to be ______________.

Write the domain of $N(s)$ in set notation.

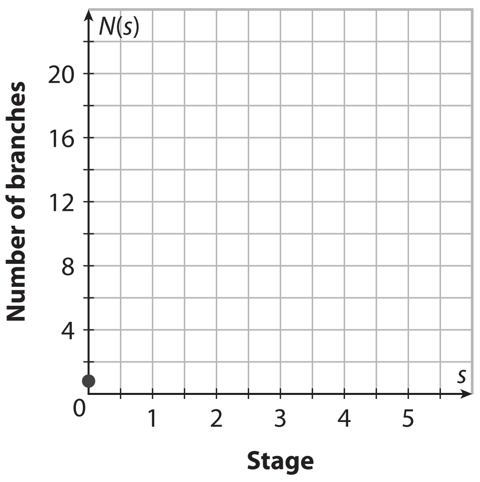
$\{s \mid s \text{ is a } \_\_\_\_\_\_ \text{ number}\}$

H. Similarly, the range of a function is the set of all possible values that the function can output over the domain. Let $N(s) = b$, the ______________________.

The range of $N(s)$ is $\{1, 2, 4, \square, \square, \square, \ldots\}$.

The range of $N(s)$ is $\{N \mid N = 2^s, \text{ where } s \text{ is } \_\_\_\_\_\_\_\_\_\_\_\_\}$.

I. Graph the first five values of $N(s)$ on the axes provided. The first value has been graphed for you.



J. As s increases, $N(s)$ ____________.

$N(s)$ is ______________ function.

Ⓚ Complete the table for branch length.

Stage Number	Number of Branches	Branch Length
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{\boxed{}}$
4	16	$\frac{1}{\boxed{}}$
5	32	$\frac{1}{\boxed{}}$
$\vdots$	$\vdots$	$\vdots$
n	2^n	$\frac{1}{\boxed{}}$

Ⓛ Write $L(s)$ expressing the branch length as a function of the Stage.

$$L(s) = \left(\frac{1}{\boxed{}}\right)^{\boxed{}}$$

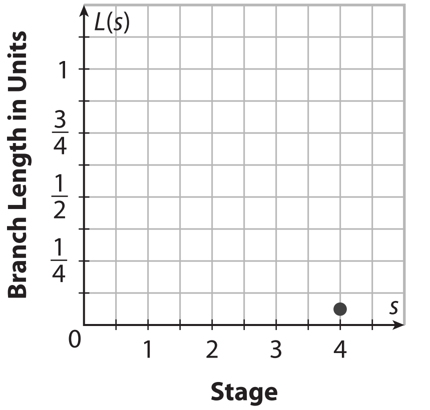
Ⓜ Write the domain and range of $L(s)$ in set notation.

The domain of $L(s)$ is $\left\{ s \mid s \text{ is a } \underline{\qquad\qquad} \text{ number} \right\}$.

The range of $L(s)$ is $\left\{ 1, \frac{1}{2}, \frac{1}{4}, \boxed{}, \boxed{}, \boxed{}, \dots \right\}$.

The range of $L(s)$ is $\left\{ L \mid L = \left(\frac{1}{2}\right)^s, \text{ where } s \text{ is } \underline{\qquad\qquad\qquad} \right\}$.

Ⓝ Graph the first five values of $L(s)$ on the axes provided. The fifth point has been graphed for you.



Ⓞ As s increases, $L(s)$ ____________. $L(s)$ is ______________ function.

Reflect

1. What is the total length added at each stage?

____

____

____

2. Is the total length of all the branches a sequence? If so, identify the sequence.

____

Explain 1 Writing Explicit and Recursive Rules for Geometric Sequences

A sequence is a set of numbers related by a common rule. All sequences start with an initial term. In a **geometric sequence**, the ratio of any term to the previous term is constant. This constant ratio is called the **common ratio** and is denoted by $r \left(r \neq 1\right)$. In the **explicit form** of the sequence, each term is found by evaluating the function $f(n) = ar^n$ or $f(n) = ar^{n-1}$ where a is the initial value and r is the common ratio, for some whole number n. Note that there are two forms of the explicit rule because it is permissible to call the initial value the first term or to call ar the first term.

A geometric sequence can also be defined recursively by $f(n) = r \cdot f(n-1)$ where either $f(0) = a$ or $f(1) = a$, again depending on the way the terms of the sequence are numbered. $f(n) = r \cdot f(n-1)$ is called the **recursive rule** for the sequence.

Example 1 **Write the explicit and recursive rules for a geometric sequence given a table of values.**

Ⓐ

n	0	1	2	3	4	$\cdots$	$j-1$	j	$\cdots$
$f(n)$	3	6	12	24	48	$\cdots$	$ar^{(j-1)}$	ar^{j}	$\cdots$

Determine a and r, then write the explicit and recursive rules.

Find the common ratio: $\dfrac{f(n)}{f(n-1)} = r.$ $\qquad \dfrac{f(1)}{f(0)} = \dfrac{6}{3} = 2 = r$

Find the initial value, $a = f(0)$, from the table. $\qquad f(0) = 3 = a$

Find the explicit rule: $f(n) = ar^n$. $\qquad f(n) = 3 \cdot (2)^n$

Write the recursive rule. $\qquad f(n) = 2 \cdot f(n-1), n \geq 1$ and $f(0) = 3$

The explicit rule is $f(n) = 3 \cdot (2)^n$ and the recursive rule is $f(n) = 2 \cdot f(n-1), n \geq 1$ and $f(0) = 3$.

Ⓑ

n	1	2	3	4	5	$\cdots$	$j-1$	j	$\cdots$
$f(n)$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	$\cdots$	$ar^{(j-1)}$	ar^{j}	$\cdots$

Determine a and r, then write the explicit and recursive rules.

Find the common ratio: $\dfrac{f(n)}{f(n-1)} = r.$ $\qquad \dfrac{f(\square)}{f(\square)} = \dfrac{\square}{\square} = \square = r$

Find the initial value, $a = f(1)$, from the table. $\qquad f(1) = \square = a$

Find the explicit rule: $f(n) = ar^{n-1}$. $\qquad f(n) = \square \cdot \left(\square\right)^{n-1}$

Write the recursive rule. $\qquad f(n) = \square \cdot f(n-1), n \geq \square$ and $f(1) = \square$

The explicit rule is $f(n) = \square$ and the recursive rule is $f(n) = \square \cdot f(n-1), n \geq \square$ where $f(1) = \square$.

Reflect

3. Discussion If you were told that a geometric sequence had an initial value of $f(5) = 5$, could you write an explicit and a recursive rule for the function? What would the explicit rule be?

Your Turn

Write the explicit and recursive rules for a geometric sequence given a table of values.

4.

n	0	1	2	3	4	5	6	...
$f(n)$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	...

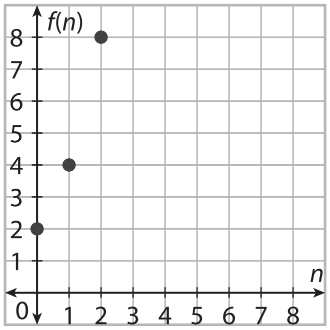
5.

n	1	2	3	4	5	6	7	...
$f(n)$	0.001	0.01	0.1	1	10	100	1000	...

Explain 2 Graphing Geometric Sequences

To graph a geometric sequence given an explicit or a recursive rule you can use the rule to generate a table of values and then graph those points on a coordinate plane. Since the domain of a geometric sequence consists only of whole numbers, its graph consists of individual points, not a smooth curve.

Example 2 **Given either an explicit or recursive rule for a geometric sequence, use a table to generate values and draw the graph of the sequence.**



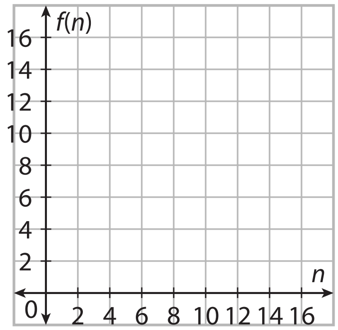
(A) Explicit rule: $f(n) = 2 \cdot 2^n,\ n \geq 0$

Use a table to generate points.

n	0	1	2	3	4	5	⋯
f(n)	2	4	8	16	32	64	⋯

Plot the first three points on the graph.

(B) Recursive rule: $f(n) = 0.5 \cdot f(n-1),\ n \geq 1$ and $f(0) = 16$



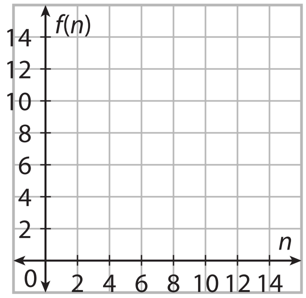
Use a table to generate points.

n	0	1	2	3				⋯
f(n)								⋯

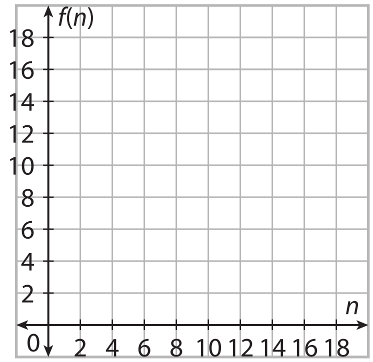
Your Turn

Given either an explicit or recursive rule for a geometric sequence, use a table to generate values and draw the graph of the sequence.

6. $f(n) = 3 \cdot 2^{n-1},\ n \geq 1$



n	1	2	3	4	5	⋯
f(n)						⋯



7. $f(n) = 3 \cdot f(n-1),\ n \geq 2$ and $f(1) = 2$

n	1	2	3	4	5	⋯
f(n)						⋯

Explain 3 Modeling With a Geometric Sequence

Given a real-world situation that can be modeled with a geometric sequence, you can use an explicit or a recursive rule to answer a question about the situation.

Example 3 **Write both an explicit and recursive rule for the geometric sequence that models the situation. Use the sequence to answer the question asked about the situation.**

(A) The Wimbledon Ladies' Singles Championship begins with 128 players. Each match, two players play and only one moves to the next round. The players compete until there is one winner. How many rounds must the winner play?

Analyze Information

Identify the important information:

- The first round requires ______ matches, so $a = \square$.
- The next round requires half as many matches, so $r = \square$.

Formulate a Plan

Let n represent the number of rounds played and let $f(n)$ represent the number of matches played at that round. Create the explicit rule and the recursive rule for the tournament. The final round will have ______ match(es), so substitute this value into the explicit rule and solve for n.

Solve

The explicit rule is $f(n) = \square$, $n \geq 1$.

The recursive rule is $f(n) = \square \cdot f(n-1)$, $n \geq 2$ and $f(1) = \square$.

The final round will have 1 match, so substitute 1 for $f(n)$ into the explicit rule and solve for n.

$$f(n) = 64 \cdot \left(\frac{1}{2}\right)^{n-1}$$
$$\square = 64 \cdot \left(\frac{1}{2}\right)^{n-1}$$
$$\square = \left(\frac{1}{2}\right)^{n-1}$$
$$\left(\frac{1}{2}\right)^{\square} = \left(\frac{1}{2}\right)^{n-1}$$

Two powers with the same positive base other than 1 are equal if and only if the exponents are equal.

$$\left(\frac{1}{2}\right)^{\square} = \left(\frac{1}{2}\right)^{n-1}$$
$$\square = n - 1$$
$$\square = n$$

The winner must play in ______ rounds.

Justify and Evaluate

The answer of 7 rounds makes sense because using the explicit rule gives

$f(7) = \boxed{}$ and the final round will have 1 match(es). This result can be checked

using the recursive rule, which again results in $f(7) = \boxed{}$.

Your Turn

Write both an explicit and recursive rule for the geometric sequence that models the situation. Use the sequence to answer the question asked about the situation.

8. A particular type of bacteria divides into two new bacteria every 20 minutes. A scientist growing the bacteria in a laboratory begins with 200 bacteria. How many bacteria are present 4 hours later?

Elaborate

9. Describe the difference between an explicit rule for a geometric sequence and a recursive rule.

10. How would you decide to use $n = 0$ or $n = 1$ as the starting value of n for a geometric sequence modeling a real-world situation?

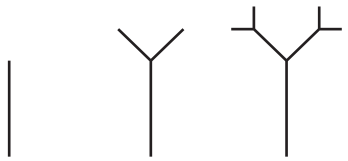
11. **Essential Question Check-In** How can you define a geometric sequence in an algebraic way? What information do you need to write these rules?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

You are creating self-similar fractal trees. You start with a trunk of length 1 unit (at Stage 0). Then the trunk splits into two branches each one-third the length of the trunk. Then each one of these branches splits into two new branches, with each branch one-third the length of the previous one.



1. Can the length of the new branches at each stage be described with a geometric sequence? Explain. If so, find the explicit form for the length of each branch.

2. Can the number of new branches at each stage be described with a geometric sequence? Explain. If so, find the recursive rule for the number of new branches.

3. Can the total length of the new branches at each stage be modeled with a geometric sequence? Explain. (The total length of the new branches is the sum of the lengths of all the new branches.)

Write the explicit and recursive rules for a geometric sequence given a table of values.

4.

n	0	1	2	3	4	...
$f(n)$	0.1	0.3	0.9	2.7	8.1	...

5.

n	0	1	2	3	4	...
$f(n)$	100	10	1	0.1	0.01	...

6.

n	1	2	3	4	5	...
$f(n)$	1000	100	10	1	0.1	...

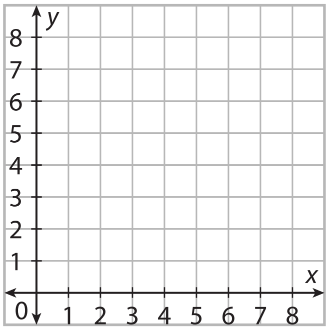
7.

n	1	2	3	4	5	...
$f(n)$	10^{50}	10^{47}	10^{44}	10^{41}	10^{38}	...

Given either an explicit or recursive rule for a geometric sequence, use a table to generate values and draw the graph of the sequence.

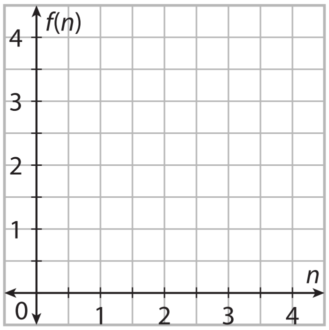
8. $f(n) = \left(\frac{1}{2}\right) \cdot 4^n,\ n \geq 0$

n	0	1	2	3	4	...
f(n)						...



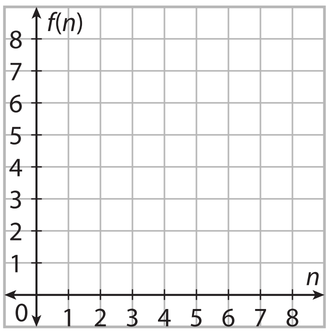
9. $f(n) = 2 \cdot f(n-1),\ n \geq 1$ and $f(0) = 0.5$

n	0	1	2	3	4	...
f(n)						...



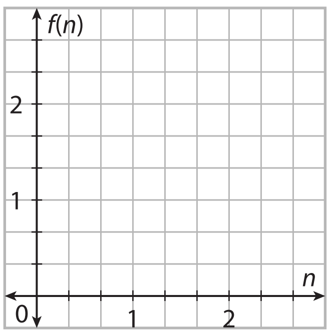
10. $f(n) = 0.5 \cdot f(n-1),\ n \geq 2$ and $f(1) = 8$

n	1	2	3	4	5	...
f(n)						...



11. $f(n) = \frac{2}{3} \cdot f(n-1),\ n \geq 2$ and $f(1) = 1$

n	1	2	3	4	5	...
f(n)						...



Write both an explicit and recursive rule for the geometric sequence that models the situation. Use the sequence to answer the question asked about the situation.

12. The Alphaville Youth Basketball committee is planning a single-elimination tournament (for all the games at each round, the winning team advances and the losing team is eliminated). The committee wants the winner to play 4 games. How many teams should the committee invite?

13. An online video game tournament begins with 1024 players. Four players play in each game. In each game there is only one winner, and only the winner advances to the next round. How many games will the winner play?

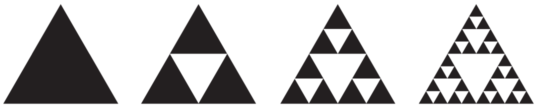
14. **Genealogy** You have 2 biological parents, 4 biological grandparents, and 8 biological great-grandparents.

a. How many direct ancestors do you have if you trace your ancestry back 6 generations? How many direct ancestors do you have if you go back 12 generations?

b. **What if…?** How does the explicit rule change if you are considered the first generation?

15. **Fractals** **Waclaw Sierpinski designed various fractals. He would take a geometric figure, shade it in, and then start removing the shading to create a fractal pattern.**

a. The Sierpinski triangle is a fractal based on a triangle. In each iteration, the center of each shaded triangle is removed.



Given that the area of the original triangle is 1 square unit, write a sequence for the area of the *n*th iteration of the Sierpinski triangle. (The first iteration is the original triangle.)

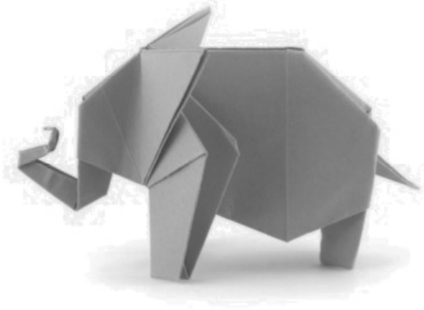
b. The Sierpinski carpet is a fractal based on a square. In each iteration, the center of each shaded square is removed.

Given that the area of the original square is 1 square unit, write a sequence for the area of the *n*th iteration of the Sierpinski carpet. (The first iteration is the original square.)

c. Find the shaded area of the fourth iteration of the Sierpinski carpet.

16. A piece of paper is 0.1 millimeter thick. When folded, the paper is twice as thick.

a. Find both the explicit and recursive rule for this geometric sequence.



b. Studies have shown that you can fold a piece of paper a maximum of 7 times. How thick will the paper be if it is folded on top of itself 7 times?

c. Assume that you could fold the paper as many times as you want. How many folds would be required for the paper to be taller than Mount Everest at 8850 meters? (*Hint:* Use a calculator to generate two large powers of 2 and check if the required number of millimeters is between those two powers. Continue to refine your guesses.)

H.O.T. Focus on Higher Order Thinking

17. Justify Reasoning Suppose you have the following table of points of a geometric sequence. The table is incomplete so you do not know the initial value. Determine whether each of the following can or cannot be the rule for the function in the table. If a function cannot be the rule for the sequence, explain why.

n	...	4	5	6	7	...
$f(n)$	...	6	12	24	48	...

A. $f(n) = 2^n$

B. $f(n) = \frac{3}{8} \cdot (2)^n$

C. $f(n) = 2 \cdot f(n-1), n \geq 1$ and $f(0) = 6$

D. $f(n) = \frac{3}{4} \cdot (2)^{n-1}$

E. $f(n) = 2 \cdot f(n-1), n \geq 1$ and $f(0) = \frac{3}{8}$

F. $f(n) = 2 \cdot f(n-1), n \geq 1$ and $f(1) = \frac{3}{4}$

G. $f(n) = (1.5) \cdot (2)^{n-2}$

H. $f(n) = 3 \cdot (2)^{n-3}$

18. Communicate Mathematical Ideas Show that the rules $f(n) = ar^n$ for $n \geq 0$ and $f(n) = ar^{n-1}$ for $n \geq 1$ for a geometric sequence are equivalent.

Lesson Performance Task

Have you ever heard of musical octaves? An octave is the interval between a musical note and the same musical note in the next higher or lower pitch. The frequencies of the sound waves of successive octaves of a note form a geometric sequence. For example, the table shows the frequencies in hertz (Hz), or cycles per second, produced by playing the note D in ascending octaves, D_0 being the lowest D note audible to the human ear.

Scale of D's	
Note	**Frequency (Hz)**
D_0	18.35
D_1	36.71
D_2	73.42
D_3	146.83

a. Explain how to write an explicit rule and a recursive rule for the frequency of D notes in hertz, where $n = 1$ represents D_1.

b. The note commonly called "middle D" is D_4. Use the explicit rule or the recursive rule from part **a** to predict the frequency for middle D.

c. Humans generally cannot hear sounds with frequencies greater than 20,000 Hz. What is the first D note that humans cannot hear? Explain.