Name ______________________ Class ____________ Date ________

12.1 Arithmetic Sequences

Essential Question: What are algebraic ways to define an arithmetic sequence?

Resource Locker

Explore Investigating Arithmetic Sequences

Consider a staircase where the vertical distance between steps is 7.5 inches and you must walk up 14 steps to get from the first floor to the second floor, a total vertical distance of 105 inches. Define two functions: $B(s)$, which models the distance from the bottom of the staircase (the first floor) to the bottom of your foot, and $T(s)$, which models the distance from the bottom of your foot to the top of the staircase (the second floor). For both functions, the independent variable s represents the number of steps that you have walked up.

(A) Complete the table. Show your calculations.

s	$B(s)$	$T(s)$
0	0	105
1	$0 + 7.5 = 7.5$	
2	$0 + 2(7.5) = 15$	
3		
4		

(B) Based on the patterns in the table, write rules for the two functions in terms of s.

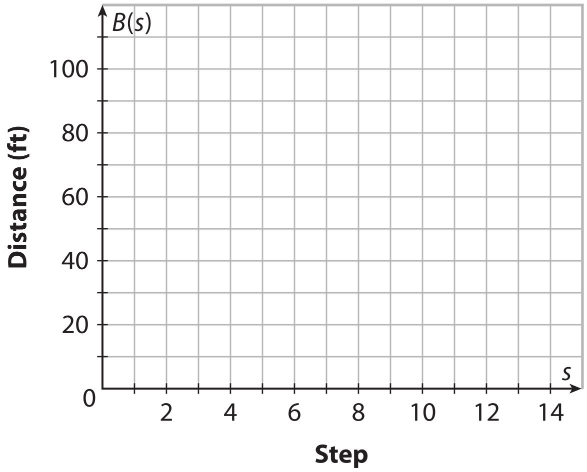
$B(s) =$ ______

$T(s) =$ ______

(C) Identify the domain and range of $B(s)$.

- The domain of $B(s)$ is $\{0, 1, 2, 3, 4, \_, \_, \_, \_, \_, \_, \_, \_, \_, \_\}$.
- The range of $B(s)$ is $\{0, 7.5, 15, \_, \_, \_, \_, \_, \_, \_, \_, \_, \_, \_, \_, \_\}$.

(D) Graph $B(s)$.

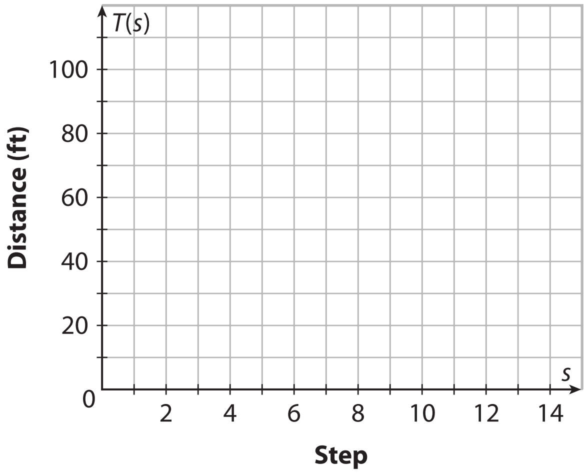


(E) Identify the domain and range of $T(s)$.

- The domain of $T(s)$ is
$\{0, 1, 2, 3, 4, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_\}$.

- The range of $T(s)$ is
$\{105, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_, \_\_\}$.

(F) Graph $T(s)$.



Reflect

1. Both $B(s)$ and $T(s)$ are linear functions, but their graphs consist of discrete points. Why?

2. How are $B(s)$ and $T(s)$ different? Why?

Explain 1 Writing Explicit and Recursive Rules for Arithmetic Sequences

A **sequence** is an ordered list of numbers. Each number in the list is called a *term* of the sequence. You can think of a sequence as a function with a subset of the set of integers as the domain and the set of terms of the sequence as the range. An **explicit rule** for a sequence defines the term in position n as a function of n. A **recursive rule** for a sequence defines the term in position n by relating it to one or more previous terms.

An **arithmetic sequence**, also known as a *discrete linear function*, is a sequence for which consecutive terms have a *common difference*. For instance, the terms of the sequence 0, 7.5, 15, 22.5, 30, 37.5, 45, 52.5, 60, 67.5, 75, 82.5, 90, 97.5, 105, which are the values of the function $B(s)$ from the Explore, have a common difference of 7.5. Likewise, the terms of the sequence 105, 97.5, 90, 82.5, 75, 67.5, 60, 52.5, 45, 37.5, 30, 22.5, 15, 7.5, 0, which are the values of the function $T(s)$ from the Explore, have a common difference of −7.5. Both sequences are arithmetic.

You can write different explicit and recursive rules for a sequence depending on what integer you use as the position number for the initial term of the sequence. The most commonly used starting position numbers are 0 and 1. The table shows rules for the sequences that you examined in the Explore.

	Sequence	
	0, 7.5, 15, 22.5, 30, 37.5, 45, 52.5, 60, 67.5, 75, 82.5, 90, 97.5, 105	105, 97.5, 90, 82.5, 75, 67.5, 60, 52.5, 45, 37.5, 30, 22.5, 15, 7.5, 0
Explicit rule when starting position is 0	$f(n) = 0 + 7.5n$ for $0 \le n \le 14$	$f(n) = 105 - 7.5n$ for $0 \le n \le 14$
Explicit rule when starting position is 1	$f(n) = 0 + 7.5(n - 1)$ for $1 \le n \le 15$	$f(n) = 105 - 7.5(n - 1)$ for $1 \le n \le 15$
Recursive rule when starting position is 0	$f(0) = 0$ and $f(n) = f(n - 1) + 7.5$ for $1 \le n \le 14$	$f(0) = 105$ and $f(n) = f(n - 1) - 7.5$ for $1 \le n \le 14$
Recursive rule when starting position is 1	$f(1) = 0$ and $f(n) = f(n - 1) + 7.5$ for $2 \le n \le 15$	$f(1) = 105$ and $f(n) = f(n - 1) - 7.5$ for $2 \le n \le 15$

In general, when 0 is the starting position for the initial term a of an arithmetic sequence with common difference d, the sequence has the explicit rule $f(n) = a + dn$ for $n \ge 0$ and the recursive rule $f(0) = a$ and $f(n) = f(n-1) + d$ for $n \ge 1$. When 1 is the starting position of the initial term, the sequence has the explicit rule $f(n) = a + d(n - 1)$ for $n \ge 1$ and the recursive rule $f(1) = a$ and $f(n) = f(n-1) + d$ for $n \ge 2$.

Example 1 **Use the given table to write an explicit and a recursive rule for the sequence.**

(A)

n	0	1	2	3	4	5
$f(n)$	2	5	8	11	14	17

First, check the differences of consecutive values of $f(n)$:

$5 - 2 = 3$, $8 - 5 = 3$, $11 - 8 = 3$, $14 - 11 = 3$, and $17 - 14 = 3$

The differences are the same, so the sequence is arithmetic.

The initial term a of the sequence is 2, and its position number is 0. As already observed, the common difference d is 3.

So, the explicit rule for the sequence is $f(n) = 2 + 3n$ for $0 \le n \le 5$. The recursive rule is $f(0) = 2$ and $f(n) = f(n - 1) + 3$ for $1 \le n \le 5$.

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n	1	2	3	4	5	6
$f(n)$	29	25	21	17	13	9

First, check the differences of consecutive values of $f(n)$:

$25 - 29 = \square$, $21 - 25 = \square$, $17 - 21 = \square$, $13 - 17 = \square$, and $9 - 13 = \square$

The differences are the same, so the sequence [is/is not] arithmetic.

The initial term a of the sequence is ______, and its position number is ______. As already observed, the common difference d is ______.

So, the explicit rule for the sequence is $f(n) = \square$ for $\square \leq n \leq \square$. The recursive rule is $f(\square) = \square$ and $f(n) = f(n-1) + \square$ for $\square \leq n \leq \square$.

Your Turn

Use the given table to write an explicit and a recursive rule for the sequence.

3.

n	0	1	2	3	4	5
$f(n)$	−7	−2	3	8	13	18

4.

n	1	2	3	4	5	6
$f(n)$	11	5	−1	−7	−13	−19

Explain 2 Graphing Arithmetic Sequences

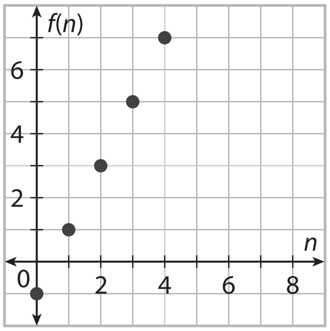
As you saw in the Explore, the graph of an arithmetic sequence consists of points that lie on a line. The arithmetic sequence 3, 7, 11, 15, 19 has a final term, so it is called a *finite* sequence and its graph has a countable number of points. The arithmetic sequence 3, 7, 11, 15, 19, … does not have a final term (indicated by the three dots), so it is called an *infinite* sequence and its graph has infinitely many points. Since you cannot show the complete graph of an infinite sequence, you should simply show as many points as the grid allows.

Example 2 **Write the terms of the given arithmetic sequence and then graph the sequence.**

(A) $f(n) = -1 + 2n$ for $0 \leq n \leq 4$

Make a table of values.

n	$f(n)$
0	$-1 + 2(0) = -1$
1	$-1 + 2(1) = 1$
2	$-1 + 2(2) = 3$
3	$-1 + 2(3) = 5$
4	$-1 + 2(4) = 7$



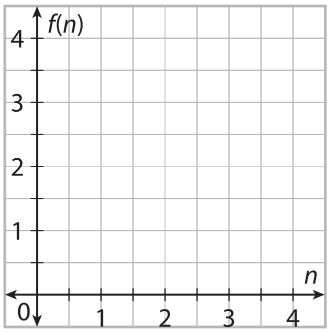
So, the sequence is −1, 1, 3, 5, 7.

Graph the sequence.

(B) $f(1) = 4$ and $f(n) = f(n-1) - 0.25$ for $n \geq 2$

Make a table of values, bearing in mind that the table could be extended because the sequence is infinite.

n	$f(n)$
1	4
2	$f(2) = f(1) - 0.25 = 4 - 0.25 = \boxed{}$
3	$f(3) = f(2) - 0.25 = \boxed{} - 0.25 = \boxed{}$
4	$f(4) = f(3) - 0.25 = \boxed{} - 0.25 = \boxed{}$
5	$f(5) = f(4) - 0.25 = \boxed{} - 0.25 = \boxed{}$



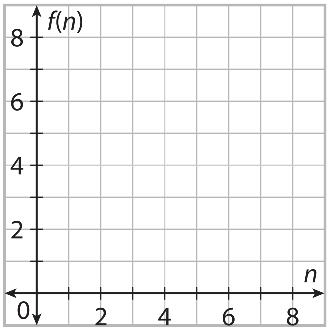
So, the sequence is ____________________.

Graph the sequence.

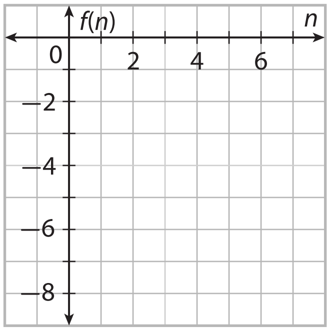
Your Turn

Write the terms of the given arithmetic sequence and then graph the sequence.

5. $f(n) = 8 - \frac{2}{3}(n - 1)$ for $1 \leq n \leq 7$



6. $f(0) = -3$ and $f(n) = f(n - 1) - 1$ for $n \geq 1$



Explain 3 Modeling with Arithmetic Sequences

Some real-world situations, like the situation in the Explore, can be modeled with an arithmetic sequence. You can then use a rule for the sequence to solve problems related to the situation.

Example 3 **Write a recursive rule and an explicit rule for an arithmetic sequence that models the situation. Then use the rule to answer the question.**

A There are 19 seats in the row nearest the stage of a theater. Each row after the first one has 2 more seats than the row before it. How many seats are in the 13th row?

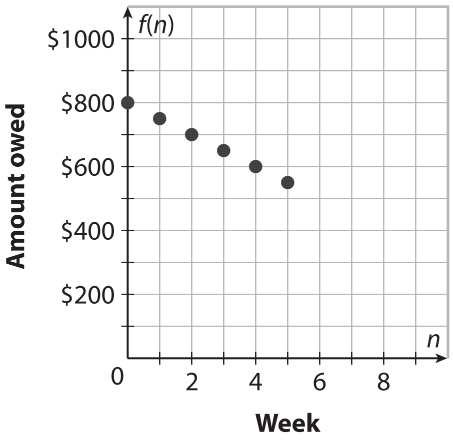
Let n represent the row number, starting with 1 for the first row. The verbal description gives you a recursive rule: $f(1) = 19$ and $f(n) = f(n - 1) + 2$ for $n \geq 2$. Since the initial term is 19 and the common difference is 2, an explicit rule is $f(n) = 19 + 2(n - 1)$ for $n \geq 1$.

To find the number of seats in the 13th row, find using the explicit rule.

$f(13) = 19 + 2(13 - 1) = 43$

So, there are 43 seats in the 13th row.

B A student with a part-time job borrowed money from her parents to purchase a bicycle. The graph shows the amount the student owes her parents as she makes equal weekly payments. The amount owed is shown only for the first 5 weeks.



In how many weeks after purchasing the bicycle will the loan be paid off?

Let n represent the number of weeks since the loan was made, starting with ____ for the time at which the loan was made. The sequence of amounts owed is arithmetic because the weekly payments are ________ . To determine the amount of the weekly payment, let $f(n)$ represent the amount owed at week n, and observe from the graph that $f(0) = \boxed{}$ and $f(2) = \boxed{}$. The general explicit rule for the sequence is $f(n) = a + dn$. Substituting 2 for n, the value of $f(0)$ for a, and the value of $f(2)$ for $f(n)$ in $f(n) = a + dn$, you can solve for the common difference d:

$$f(2) = f(0) + 2d$$

$$\boxed{} = \boxed{} + 2d$$

$$\boxed{} = 2d$$

$$\boxed{} = d$$

So, the amount of the weekly payment is ____ , and an explicit rule for the sequence is $f(n) = \boxed{} + \left(\boxed{}\right)n$. A recursive rule for the sequence is $f(0) = \boxed{}$ and $f(n) = f(n-1) + \boxed{}$ for $n \geq 1$.

To determine when the loan will be paid off, you want to find the value of n for which $f(n) = 0$. Use the explicit rule.

$$f(n) = 0$$

$$\boxed{} + \left(\boxed{}\right)n = 0$$

$$\boxed{}\,n = \boxed{}$$

$$n = \boxed{}$$

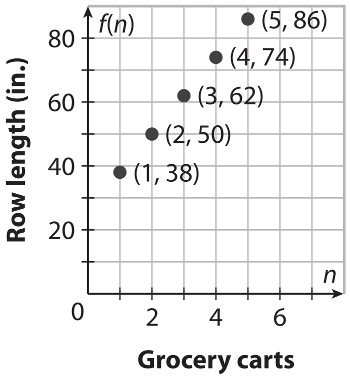
So, the loan will be paid off in ____ weeks.

Your Turn

Write a recursive rule and an explicit rule for an arithmetic sequence that models the situation. Then use the rule to answer the question.

7. The starting salary for a summer camp counselor is \$395 per week. In each of the subsequent weeks, the salary increases by \$45 to encourage experienced counselors to work for the entire summer. If the salary is \$710 for the last week of the camp, for how many weeks does the camp run?

8. The graph shows the length, in inches, of a row of grocery carts when various numbers of carts are nested together. What is the length of a row of 25 nested carts?



Elaborate

9. **Discussion** Is it easier to use an explicit rule or a recursive rule to find the 10th term in an arithmetic sequence? Explain.

10. What do you know about the terms in an arithmetic sequence with a common difference of 0?

11. Describe the difference between the graph of an arithmetic sequence with a positive common difference and the graph of an arithmetic sequence with a negative common difference.

12. **Essential Question Check-In** Does the rule $f(n) = -2 + 5n$ for $n \geq 0$ define an arithmetic sequence, and is the rule explicit or recursive? How do you know?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. Consider the staircase in the Explore. How would the functions $B(s)$ and $T(s)$ change if the staircase were a spiral staircase going from the first floor to the third floor, with the same step height and distance between floors?

Use the given table to write an explicit and a recursive rule for the sequence.

2.

n	0	1	2	3	4
$f(n)$	−6	1	8	15	22

3.

n	0	1	2	3	4
$f(n)$	8	5	2	−1	−4

Given the recursive rule for an arithmetic sequence, write the explicit rule.

4. $f(0) = 6$ and $f(n) = f(n - 1) + 5$ for $n \geq 1$

5. $f(1) = 19$ and $f(n) = f(n - 1) - 10$ for $n \geq 2$

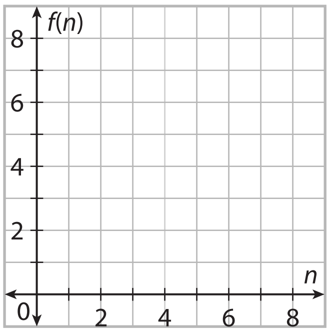
Given the explicit rule for an arithmetic sequence, write the recursive rule.

6. $f(n) = 9.6 - 0.2(n - 1)$ for $n \geq 1$

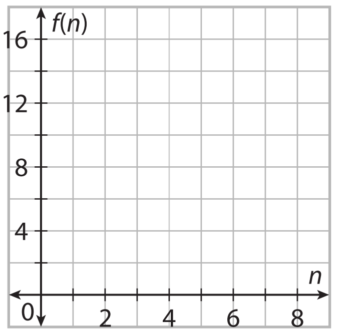
7. $f(n) = 14 + 8n$ for $n \geq 0$

Write the terms of the given arithmetic sequence and then graph the sequence.

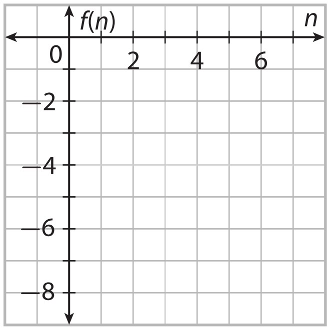
8. $f(n) = 7 - \frac{1}{2}n$ for $n \geq 0$



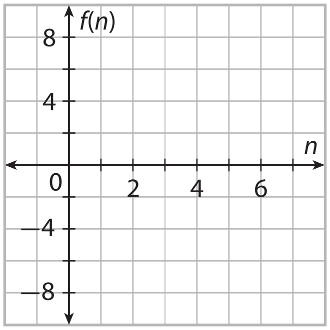
9. $f(n) = 3 + 2(n - 1)$ for $1 \leq n \leq 5$



10. $f(1) = -0.5$ and $f(n) = f(n-1) - 0.5$ for $n \geq 2$



11. $f(0) = -5$ and $f(n) = f(n-1) + 3$ for $1 \leq n \leq 4$



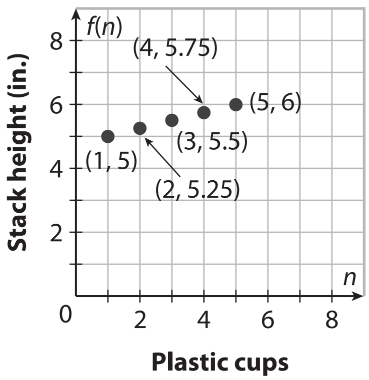
Write a recursive rule and an explicit rule for an arithmetic sequence that models the situation. Then use the rule to answer the question.

12. Thomas begins an exercise routine for 20 minutes each day. Each week he plans to add 5 minutes per day to the length of his exercise routine. For how many minutes will he exercise each day of the 6th week?

13. The Louvre pyramid in Paris, France, is built of glass panes. There are 4 panes in the top row, and each additional row has 4 more panes than the previous row. How many panes are in the 17th row?

14. Clarissa is buying a prom dress on layaway. The dress costs \$185. She makes a down payment of \$20 to put the dress on layaway and then makes weekly payments of \$15. In how many weeks is the dress paid off?

15. The graph shows the height, in inches, of a stack of various numbers of identical plastic cups. The stack of cups will be placed on a shelf with 12 inches of vertical clearance with the shelf above. What number of cups can be in the stack without having a tight fit?



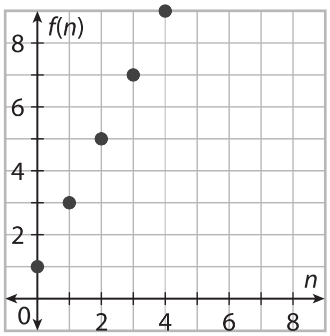
16. Determine whether or not each of the following sequences is arithmetic. Select the correct response for each lettered part.

A. 1, 1, 2, 3, 5, 8, 13, 21, 34	☐ Arithmetic	☐ Not arithmetic
B. 1, 4, 7, 10, 13, 16, 19	☐ Arithmetic	☐ Not arithmetic
C. 1, 2, 4, 9, 16, 25	☐ Arithmetic	☐ Not arithmetic
D. $-4, 3, 10, 17, 24, 31$	☐ Arithmetic	☐ Not arithmetic
E. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$	☐ Arithmetic	☐ Not arithmetic
F. $18.5, 13, 7.5, 2, -3.5, -9$	☐ Arithmetic	☐ Not arithmetic

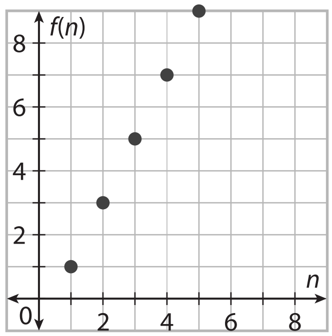
H.O.T. Focus on Higher Order Thinking

17. Multiple Representations The graphs of two arithmetic sequences are shown.

Graph of sequence A



Graph of sequence B



a. Are the sequences the same or different? Explain.

b. Write an explicit rule for each sequence.

c. How do the explicit rules indicate the geometric relationship between the two graphs?

18. Communicate Mathematical Ideas You know that if $\left(x_1, f(x_1)\right)$ and $\left(x_2, f(x_2)\right)$ are two points on the graph of a linear function, the slope of the function's graph is $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Suppose $\left(n_1, f(n_1)\right)$ and $\left(n_2, f(n_2)\right)$ are two points on the graph of an arithmetic sequence with the explicit rule $f(n) = a + dn$. What does the expression $\frac{f(n_2) - f(n_1)}{n_2 - n_1}$ tell you about the arithmetic sequence? Justify your answer.

19. Construct Arguments Show how the recursive rule $f(0) = a$ and $f(n) = f(n-1) + d$ for $n \geq 1$ generates the explicit rule $f(n) = a + dn$ for $n \geq 0$.

Lesson Performance Task

The graph shows how the cost of a personal transporter tour depends on the number of participants. Write explicit and recursive rules for the cost of the tour. Then calculate the cost of the tour for 12 participants.