Name ____________________ Class ____________ Date ________

11.2 Simplifying Radical Expressions

Essential Question: How can you simplify expressions containing rational exponents or radicals involving *n*th roots?

Resource Locker

Explore Establishing the Properties of Rational Exponents

In previous courses, you have used properties of integer exponents to simplify and evaluate expressions, as shown here for a few simple examples:

$4^2 \cdot 4^3 = 4^{2+3} = 4^5 = 1024$

$(4 \cdot x)^2 = 4^2 \cdot x^2 = 16x^2$

$\left(4^2\right)^3 = 4^{2 \cdot 3} = 4^6 = 4096$

$\frac{4^2}{4^3} = 4^{2-3} = 4^{-1} = \frac{1}{4}$

$\left(\frac{4}{x}\right)^3 = \frac{4^3}{x^3} = \frac{64}{x^3}$

Now that you have been introduced to expressions involving rational exponents, you can explore the properties that apply to simplifying them.

(A) Let $a = 64$, $b = 4$, $m = \frac{1}{3}$, and $n = \frac{3}{2}$. Evaluate each expression by substituting and applying exponents individually, as shown.

Expression	Substitute	Simplify	Result
$a^m \cdot a^n$	$64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$	$4 \cdot 512$	2048
$(a \cdot b)^n$	$(64 \cdot 4)^{\frac{3}{2}}$	$256^{\frac{3}{2}}$	
$(a^m)^n$			
$\frac{a^n}{a^m}$			
$\left(\frac{a}{b}\right)^n$			

B Complete the table again. This time, however, apply the rule of exponents that you would use for integer exponents.

Expression	Apply Rule and Substitute	Simplify	Result
$a^m \cdot a^n$	$64^{\frac{1}{3}+\frac{3}{2}}$	$64^{\frac{11}{6}}$	
$(a \cdot b)^n$			
$(a^m)^n$			
$\frac{a^n}{a^m}$			
$\left(\frac{a}{b}\right)^n$			

Reflect

1. Compare your results in Steps A and B. What can you conclude?

2. In Steps A and B, you evaluated $\frac{a^n}{a^m}$ two ways. Now evaluate $\frac{a^m}{a^n}$ two ways, using the definition of negative exponents. Are your results consistent with your previous conclusions about integer and rational exponents?

Explain 1 Simplifying Rational-Exponent Expressions

Rational exponents have the same properties as integer exponents.

Properties of Rational Exponents

For all nonzero real numbers *a* and *b* and rational numbers *m* and *n*

Words	Numbers	Algebra
Product of Powers Property To multiply powers with the same base, add the exponents.	$12^{\frac{1}{2}} \cdot 12^{\frac{3}{2}} = 12^{\frac{1}{2}+\frac{3}{2}} = 12^2 = 144$	$a^m \cdot a^n = a^m + n$
Quotient of Powers Property To divide powers with the same base, subtract the exponents.	$\frac{125^{\frac{2}{3}}}{125^{\frac{1}{3}}} = 125^{\frac{2}{3}-\frac{1}{3}} = 125^{\frac{1}{3}} = 5$	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power Property To raise one power to another, multiply the exponents.	$\left(8^{\frac{2}{3}}\right)^3 = 8^{\frac{2}{3}\cdot 3} = 8^2 = 64$	$(a^m)^n = a^{m \cdot n}$
Power of a Product Property To find a power of a product, distribute the exponent.	$(16 \cdot 25)^{\frac{1}{2}} = 16^{\frac{1}{2}} \cdot 25^{\frac{1}{2}} = 4 \cdot 5 = 20$	$(ab)^m = a^m b^m$
Power of a Quotient Property To find the power of a qoutient, distribute the exponent.	$\left(\frac{16}{81}\right)^{\frac{1}{4}} = \frac{16^{\frac{1}{4}}}{81^{\frac{1}{4}}} = \frac{2}{3}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example 1 **Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.**

A **a.** $25^{\frac{3}{5}} \cdot 25^{\frac{7}{5}}$

$= 25^{\frac{3}{5}+\frac{7}{5}}$ Product of Powers Prop.

$= 25^2$ Simplify.

$= 625$

b. $\frac{8^{\frac{1}{3}}}{8^{\frac{2}{3}}}$

$= 8^{\frac{1}{3}-\frac{2}{3}}$ Quotient of Powes Prop.

$= 8^{-\frac{1}{3}}$ Simplify.

$= \frac{1}{8^{\frac{1}{3}}}$ Definition of neg. power

$= \frac{1}{2}$ Simplify.

(B) **a.** $\left(\dfrac{y^{\frac{4}{3}}}{16y^{\frac{2}{3}}}\right)^{\frac{3}{2}}$

$= \left(\dfrac{y^{\frac{4}{3}-\frac{2}{3}}}{16}\right)^{\frac{3}{2}}$ $\boxed{}$ Prop.

$= \left(\dfrac{\boxed{}}{16}\right)^{\frac{3}{2}}$ Simplify.

$= \dfrac{\left(y^{\frac{2}{3}}\right)^{\frac{3}{2}}}{16^{\frac{3}{2}}}$ $\boxed{}$ Prop.

$= \dfrac{y^{\frac{2}{3}\cdot\frac{3}{2}}}{16^{\frac{3}{2}}}$ $\boxed{}$ Prop.

$= \boxed{}$ Simplify.

b. $\left(27x^{\frac{3}{4}}\right)^{\frac{2}{3}}$

$= \boxed{}^{\frac{2}{3}}\left(\boxed{}\right)^{\frac{2}{3}}$ Power of a Product Prop.

$= 27^{\frac{2}{3}}\left(x^{\boxed{}}\right)$ Power of a Power Prop.

$= \boxed{}$ Simplify.

Your Turn

Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.

3. $\left(12^{\frac{2}{3}} \cdot 12^{\frac{4}{3}}\right)^{\frac{3}{2}}$

4. $\dfrac{\left(6x^{\frac{1}{3}}\right)^2}{x^{\frac{5}{3}}y}$

Explain 2 Simplifying Radical Expressions Using the Properties of Exponents

When you are working with radical expressions involving *n*th roots, you can rewrite the expressions using rational exponents and then simplify them using the properties of exponents.

Example 2 **Simplify the expression by writing it using rational exponents and then using the properties of rational exponents. Assume that all variables are positive. Exponents in simplified form should all be positive.**

(A) $x\left(\sqrt[3]{2y}\right)\left(\sqrt[3]{4x^2y^2}\right)$

$= x(2y)^{\frac{1}{3}}\left(4x^2y^2\right)^{\frac{1}{3}}$ Write using rational exponents.

$= x\left(2y \cdot 4x^2y^2\right)^{\frac{1}{3}}$ Power of a Product Property

$= x\left(8x^2y^3\right)^{\frac{1}{3}}$ Product of Powers Property

$= x\left(2x^{\frac{2}{3}}y\right)$ Power of a Product Property

$= 2x^{\frac{5}{3}}y$ Product of Powers Property

Ⓑ $\dfrac{\sqrt{64y}}{\sqrt[3]{64y}}$

$= \dfrac{(64y)^{\frac{1}{2}}}{(64y)^{\frac{1}{3}}}$ Write using rational exponents.

$= (64y)^{\square}$ Quotient of Powers Property

$= (64y)^{\square}$ Simplify.

$= \square$ Power of a Product Property

$= \square$ Simplify.

Your Turn

Simplify the expression by writing it using rational exponents and then using the properties of rational exponents.

5. $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}}$

6. $\sqrt[6]{16^3} \cdot \sqrt[4]{4^6} \cdot \sqrt[3]{8^2}$

Explain 3 Simplifying Radical Expressions Using the Properties of n^{th} Roots

From working with square roots, you know, for example, that $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$ and $\dfrac{\sqrt{8}}{\sqrt{2}} \cdot = \sqrt{\dfrac{8}{2}} = \sqrt{4} = 2$. The corresponding properties also apply to *n*th roots.

Properties of *n*th Roots

For a > 0 and b > 0

Words	Numbers	Algebra
Product Property of Roots The *n*th root of a product is equal to the product of the *n*th roots.	$\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
Quotient Property of Roots The *n*th root of a Quotient is equal to the Quotient of the *n*th roots.	$\sqrt{\dfrac{25}{16}} = \dfrac{\sqrt{25}}{\sqrt{16}} = \dfrac{5}{4}$	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Example 3 **Simplify the expression using the properties of *n*th roots. Assume that all variables are positive. Rationalize any irrational denominators.**

(A) $\sqrt[3]{256x^3y^7}$

$\sqrt[3]{256x^3y^7}$

$= \sqrt[3]{2^8 \cdot x^3y^7}$ Write 256 as a power.

$= \sqrt[3]{2^6 \cdot x^3y^6} \cdot \sqrt[3]{2^2 \cdot y}$ Product Property of Roots

$= \sqrt[3]{2^6} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^6} \cdot \sqrt[3]{4y}$ Factor out perfect cubes.

$= 4xy^2\sqrt[3]{4y}$ Simplify.

(B) $\sqrt[4]{\dfrac{81}{x}}$

$\sqrt[4]{\dfrac{81}{x}}$

$= \dfrac{\sqrt[4]{81}}{\sqrt[4]{x}}$ $\boxed{}$

$= \dfrac{\boxed{}}{\sqrt[4]{x}}$ Simplify.

$= \dfrac{3}{\sqrt[4]{x}} \cdot \dfrac{\boxed{}}{\sqrt[4]{x^3}}$ Rationalize the denominator.

$= \dfrac{3\sqrt[4]{x^3}}{\sqrt[4]{x^4}}$ $\boxed{}$

$= \boxed{}$ Simplify.

Reflect

7. In Part B, why was $\sqrt[4]{x^3}$ used when rationalizing the denominator? What factor would you use to rationalize a denominator of $\sqrt[5]{4y^3}$?

Your Turn

Simplify the expression using the properties of *n*th roots. Assume that all variables are positive.

8. $\sqrt[3]{216x^{12}y^{15}}$

9. $\sqrt[4]{\dfrac{16}{x^{14}}}$

Explain 4 Rewriting a Radical-Function Model

When you find or apply a function model involving rational powers or radicals, you can use the properties in this lesson to help you find a simpler expression for the model.

(A) **Manufacturing** A can that is twice as tall as its radius has the minimum surface area for the volume it contains. The formula $S = 6\pi\left(\dfrac{V}{2\pi}\right)^{\frac{2}{3}}$ expresses the surface area of a can with this shape in terms of its volume.

a. Use the properties of rational exponents to simplify the expression for the surface area. Then write the approximate model with the coefficient rounded to the nearest hundredth.

b. Graph the model using a graphing calculator. What is the surface area in square centimeters for a can with a volume of 440 cm^3?

a. $S = 6\pi\left(\dfrac{V}{2\pi}\right)^{\frac{2}{3}}$

Power of a Quotient Property $= 6\pi \cdot \dfrac{V^{\frac{2}{3}}}{(2\pi)^{\frac{2}{3}}}$

Group Powers of 2π. $= \dfrac{3(2\pi)}{(2\pi)^{\frac{2}{3}}} \cdot V^{\frac{2}{3}}$

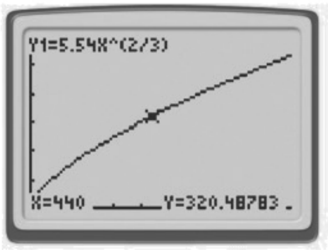
Quotient of Powers Property $= 3(2\pi)^{1-\frac{2}{3}} \cdot V^{\frac{2}{3}}$

Simplify. $= 3(2\pi)^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$

Use a calculator. $\approx 5.54V^{\frac{2}{3}}$

A simplified model is $S = 3(2\pi)^{\frac{1}{3}} \cdot V^{\frac{2}{3}}$, which gives $S \approx 5.54V^{\frac{2}{3}}$.

b.



The surface area is about 320 cm^2.

(B) **Commercial fishing** The buoyancy of a fishing float in water depends on the volume of air it contains. The radius of a spherical float as a function of its volume is given by $r = \sqrt[3]{\frac{3V}{4\pi}}$.

a. Use the properties of roots to rewrite the expression for the radius as the product of a coefficient term and a variable term. Then write the approximate formula with the coefficient rounded to the nearest hundredth.

b. What should the radius be for a float that needs to contain 4.4 ft^3 of air to have the proper buoyancy?

a. $r = \sqrt[3]{\frac{3V}{4\pi}}$

Rewrite radicand. $= \sqrt[3]{\frac{3}{4\pi} \cdot \boxed{}}$

Product Property of Roots $= \sqrt[3]{\frac{3}{4\pi}} \cdot \boxed{}$

Use a calculator $\approx \boxed{}$

The rewritten formula is $r = \boxed{}$, which gives $r \approx \boxed{}$.

b. Substitute 4.4 for V.

$r = 0.62\sqrt[3]{4.4} \approx \boxed{}$

The radius is about ____ feet.

Reflect

10. Discussion What are some reasons you might want to rewrite an expression involving radicals into an expression involving rational exponents?

Your Turn

11. The surface area as a function of volume for a box with a square base and a height that is twice the side length of the base is $S = 10\left(\frac{V}{2}\right)^{\frac{2}{3}}$. Use the properties of rational exponents to simplify the expression for the surface area so that no fractions are involved. Then write the approximate model with the coefficient rounded to the nearest hundredth.

Elaborate

12. In problems with a radical in the denominator, you rationalized the denominator to remove the radical. What can you do to remove a rational exponent from the denominator? Explain by giving an example.

13. Show why $\sqrt[n]{a^n}$ is equal to a for all natural numbers a and n using the definition of nth roots and using rational exponents.

14. Show that the Product Property of Roots is true using rational exponents.

15. **Essential Question Check-In** Describe the difference between applying the Power of a Power Property and applying the Power of a Product Property for rational exponents using an example that involves both properties.

Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

Simplify the expression. Assume that all variables are positive. Exponents in simplified form should all be positive.

1. $\left(\left(\frac{1}{16}\right)^{-\frac{2}{3}}\right)^{\frac{3}{4}}$

2. $\frac{x^{\frac{1}{3}} \cdot x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$

3. $\frac{9^{\frac{3}{2}} \cdot 9^{\frac{1}{2}}}{9^{-2}}$

4. $\left(\frac{16^{\frac{5}{3}}}{16^{\frac{5}{6}}}\right)^{\frac{9}{5}}$

5. $\frac{2xy}{\left(x^{\frac{1}{3}}y^{\frac{2}{3}}\right)^{\frac{3}{2}}}$

6. $\frac{3y^{\frac{3}{4}}}{2xy^{\frac{3}{2}}}$

Simplify the expression by writing it using rational exponents and then using the properties of rational exponents. Assume that all variables are positive. Exponents in simplified form should all be positive.

7. $\sqrt[4]{25} \cdot \sqrt[3]{5}$

8. $\frac{\sqrt[4]{2^{-2}}}{\sqrt[6]{2^{-9}}}$

9. $\frac{\sqrt[4]{3^3} \cdot \sqrt[3]{x^2}}{\sqrt{3x}}$

10. $\frac{\sqrt[4]{x^4y^6} \cdot \sqrt{x^6}}{y}$

11. $\dfrac{\sqrt[6]{s^4t^9}}{\sqrt[3]{st}}$

12. $\sqrt[4]{27} \cdot \sqrt{3} \cdot \sqrt[6]{81^3}$

Simplify the expression using the properties of *n*th roots. Assume that all variables are positive. Rationalize any irrational denominators.

13. $\dfrac{\sqrt[4]{36} \cdot \sqrt[4]{216}}{\sqrt[4]{6}}$

14. $\sqrt[4]{4096x^6y^8}$

15. $\dfrac{\sqrt[3]{x^8y^4}}{\sqrt[3]{x^2y}}$

16. $\sqrt[5]{\dfrac{125}{w^6}} \cdot \sqrt[5]{25v}$

17. **Weather** The volume of a sphere as a function of its surface area is given by $V = \dfrac{4\pi}{3}\left(\dfrac{S}{4\pi}\right)^{\frac{3}{2}}$.

a. Use the properties of roots to rewrite the expression for the volume as the product of a simplified coefficient term (with positive exponents) and a variable term. Then write the approximate formula with the coefficient rounded to the nearest thousandth.

b. A spherical weather balloon has a surface area of 500 ft^2. What is the approximate volume of the balloon?

18. **Amusement parks** An amusement park has a ride with a free fall of 128 feet. The formula $t = \sqrt{\frac{2d}{g}}$ gives the time t in seconds it takes the ride to fall a distance of d feet. The formula $v = \sqrt{2gd}$ gives the velocity v in feet per second after the ride has fallen d feet. The letter g represents the gravitational constant.

a. Rewrite each formula so that the variable d is isolated. Then simplify each formula using the fact that $g \approx 32$ ft/s^2.

b. Find the time it takes the ride to fall halfway and its velocity at that time. Then find the time and velocity for the full drop.

c. What is the ratio of the time it takes for the whole drop to the time it takes for the first half? What is the ratio of the velocity after the second half of the drop to the velocity after the first half? What do you notice?

19. Which choice(s) is/are equivalent to $\sqrt{2}$?

A. $\left(\sqrt[8]{2}\right)^4$

B. $\dfrac{2^3}{2^{-\frac{5}{2}}}$

C. $\left(4^{\frac{2}{3}} \cdot 2^{\frac{2}{3}}\right)^{\frac{1}{4}}$

D. $\dfrac{\sqrt[3]{2^2}}{\sqrt[6]{2}}$

E. $\dfrac{\sqrt{2^{-\frac{3}{4}}}}{\sqrt{2^{-\frac{7}{4}}}}$

20. **Home Heating** A propane storage tank for a home is shaped like a cylinder with hemispherical ends, and a cylindrical portion length that is 4 times the radius.

The formula $S = 12\pi \left(\frac{3V}{16\pi}\right)^{\frac{2}{3}}$ expresses the surface area of a tank with this shape in terms of its volume.

a. Use the properties of rational exponents to rewrite the expression for the surface area so that the variable V is isolated. Then write the approximate model with the coefficient rounded to the nearest hundredth.

b. Graph the model using a graphing calculator. What is the surface area in square feet for a tank with a volume of 150 ft^3 ?

H.O.T. Focus on Higher Order Thinking

21. **Critique Reasoning** Aaron's work in simplifying an expression is shown. What mistake(s) did Aaron make? Show the correct simplification.

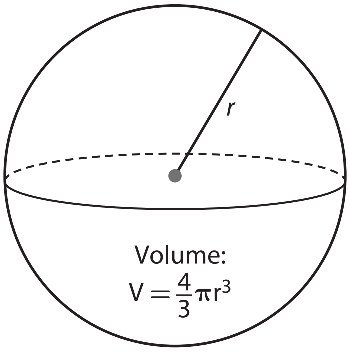
$$625^{-\frac{1}{3}} \div 625^{-\frac{4}{3}}$$
$$= 625^{-\frac{1}{3} - \left(-\frac{4}{3}\right)}$$
$$= 625^{-\frac{1}{3}\left(-\frac{3}{4}\right)}$$
$$= 625^{\frac{1}{4}}$$
$$= 5$$

22. **Critical Thinking** Use the definition of nth root to show that the Product Property of Roots is true, that is, that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$. (Hint: Begin by letting x be the nth root of a and letting y be the nth root of b.)

23. **Critical Thinking** For what real values of a is $\sqrt[4]{a}$ greater than a? For what real values of a is $\sqrt[5]{a}$ greater than a?

Lesson Performance Task

You've been asked to help decorate for a school dance, and the theme chosen is "The Solar System." The plan is to have a bunch of papier-mâché spheres serve as models of the planets, and your job is to paint them. All you're told are the volumes of the individual spheres, but you need to know their surface areas so you can get be sure to get enough paint. How can you write a simplified equation using rational exponents for the surface area of a sphere in terms of its volume?



(The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the formula for the surface area of a sphere is $A = 4\pi r^2$.)