11.1 Radical Expressions and Rational Exponents

Essential Question: How are rational exponents related to radicals and roots?

⊘ Explore Defining Rational Exponents in Terms of Roots

Remember that a number a is an nth root of a number b if $a^n = b$. As you know, a square root is indicated by $\sqrt{}$ and a cube root by $\sqrt[3]{}$. In general, the nth root of a real number a is indicated by $\sqrt[n]{a}$, where n is the **index** of the radical and a is the radicand. (Note that when a number has more than one real root, the radical sign indicates only the principal, or positive, root.)

A *rational exponent* is an exponent that can be expressed as $\frac{m}{n}$, where m is an integer and n is a natural number. You can use the definition of a root and properties of equality and exponents to explore how to express roots using rational exponents.

(A) How can you express a square root using an exponent? That is, if $\sqrt{a} = a^m$, what is m?

Given	$\sqrt{a} = a^m$
Square both sides.	$\left(\sqrt{a}\right)^2 = \left(a^m\right)^2$
Definition of square root	$\boxed{} = \left(a^m\right)^2$
Power of a power property	$a = a^{\boxed{}}$
Definition of first power	$a^{\boxed{}} = a^{2m}$
The bases are the same, so equate exponents.	$\boxed{} = \boxed{}$
Solve.	$m = \boxed{}$
So,	$\sqrt{a} = a^{\boxed{}}$.

(B) How can you express a cube root using an exponent? That is, if $\sqrt[3]{a} = a^m$, what is m?

Given	$\sqrt[3]{a} = a^m$
Cube both sides.	$\left(\sqrt{a}\right)^3 = \left(a^m\right)^3$
Definition of cube root	$\boxed{} = \boxed{}$
Power of a power property	$\boxed{} = \boxed{}$

Definition of first power $\boxed{} = \boxed{}$

The bases are the same, so equate exponents. $\boxed{} = \boxed{}$

Solve. $m = \boxed{}$

So, $\sqrt[3]{a} = a^{\boxed{}}$.

1. **Discussion** Examine the reasoning in Steps A and B. Can you apply the same reasoning for any nth root, $\sqrt[n]{a}$, where n is a natural number? Explain. What can you conclude?

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2. For a positive number a, under what condition on n will there be only one real nth root? two real nth roots? Explain.

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3. For a negative number a, under what condition on n will there be no real nth roots? one real nth root? Explain.

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⏺ Explain 1　Translating Between Radical Expressions and Rational Exponents

In the Explore, you found that a rational exponent $\frac{m}{n}$ with $m = 1$ represents an nth root, or that $a^{\frac{1}{n}} = \sqrt[n]{a}$ for positive values of a. This is also true for negative values of a when the index is odd. When $m \neq 1$, you can think of the numerator m as the power and the denominator n as the root. The following ways of expressing the exponent $\frac{m}{n}$ are equivalent.

Rational Exponents

For any natural number n, integer m, and real number a when the nth root of a is real:

Words	Numbers	Algebra
The exponent $\frac{m}{n}$ indicates the mth power of the nth root of a quantity.	$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$
The exponent $\frac{m}{n}$ indicates the nth root of the mth power of a quantity.	$4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$	$a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Notice that you can evaluate each example in the "Numbers" column using the equivalent definition.
$$27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9 \qquad 4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$$

Example 1　Translate radical expressions into expressions with rational exponents, and vice versa. Simplify numerical expressions when possible. Assume all variables are positive.

(A)　**a.** $(-125)^{\frac{4}{3}}$　　**b.** $x^{\frac{11}{8}}$　　**c.** $\sqrt[5]{6^4}$　　**d.** $\sqrt[4]{x^3}$

a. $(-125)^{\frac{4}{3}} = \left(\sqrt[3]{-125}\right)^4 = (-5)^4 = 625$

b. $x^{11/8} = \sqrt[8]{x^{11}}$ or $\left(\sqrt[8]{x}\right)^{11}$

c. $\sqrt[5]{6^4} = 6^{\frac{4}{5}}$

d. $\sqrt[4]{x^3} = x^{\frac{3}{4}}$

(B)　**a.** $\left(\dfrac{81}{16}\right)^{\frac{3}{4}}$　　**b.** $(xy)^{\frac{5}{3}}$　　**c.** $\sqrt[3]{11^6}$　　**d.** $\sqrt[3]{\left(\dfrac{2x}{y}\right)^5}$

a. $\left(\dfrac{81}{16}\right)^{\frac{3}{4}} = \left(\sqrt[\Box]{\dfrac{81}{16}}\right)^{\Box} = \left(\Box\right)^3 = \Box$

b. $(xy)^{\frac{5}{3}} = \sqrt[\Box]{(xy)^{\Box}}$ or $\left(\sqrt[\Box]{xy}\right)^{\Box}$

c. $\sqrt[3]{11^6} = 11^{\Box} = 11^{\Box} = \Box$

d. $\sqrt[3]{\left(\dfrac{2x}{y}\right)^5} = \left(\dfrac{2x}{y}\right)^{\Box}$

4. How can you use a calculator to show that evaluating $0.001728^{\frac{4}{3}}$ as a power of a root and as a root of a power are equivalent methods?

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Your Turn

5. Translate radical expressions into expressions with rational exponents, and vice versa. Simplify numerical expressions when possible. Assume all variables are positive.

a. $\left(-\dfrac{32}{243}\right)^{\frac{2}{5}}$

b. $(3y)^{\frac{b}{c}}$

c. $\sqrt[3]{0.5^9}$

d. $\left(\sqrt[u]{st}\right)^v$

⚙ Explain 2   Modeling with Power Functions

The following functions all involve a given power of a variable.

$A = \pi r^2$ (area of a circle)

$V = \frac{4}{3}\pi r^3$ (volume of a sphere)

$T = 1.11 \cdot L^{\frac{1}{2}}$ (the time T in seconds for a pendulum of length L feet to complete one back-and-forth swing)

These are all examples of *power functions*. A power function has the form $y = ax^b$ where a is a real number and b is a rational number.

**Example 2**   **Solve each problem by modeling with power functions.**

Ⓐ **Biology** The function $R = 73.3\sqrt[4]{M^3}$, known as Kleiber's law, relates the basal metabolic rate R in Calories per day burned and the body mass M of a mammal in kilograms. The table shows typical body masses for some members of the cat family.

Typical Body Mass	
Animal	**Mass (kg)**
House cat	4.5
Cheetah	55
Lion	170

a. Rewrite the formula with a rational exponent.

b. What is the value of R for a cheetah to the nearest 50 Calories?

c. From the table, the mass of the lion is about 38 times that of the house cat. Is the lion's metabolic rate more or less than 38 times the cat's rate? Explain.


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73.3*55^(3/4)
          1480.389485
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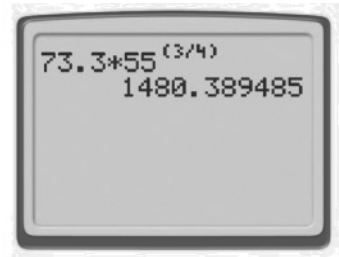
a. Because $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, $\sqrt[4]{M^3} = M^{\frac{3}{4}}$, so the formula is $R = 73.3M^{\frac{3}{4}}$.

b. Substitute 55 for M in the formula and use a calculator.

The cheetah's metabolic rate is about 1500 Calories.

c. Less; find the ratio of R for the lion to R for the house cat.

$$\frac{73.3(170)^{\frac{3}{4}}}{73.3(4.5)^{\frac{3}{4}}} = \frac{170^{\frac{3}{4}}}{4.5^{\frac{3}{4}}} \approx \frac{47.1}{3.1} \approx 15$$

The metabolic rate for the lion is only about 15 times that of the house cat.

B The function $h(m) = 241m^{-\frac{1}{4}}$ models an animal's approximate resting heart rate h in beats per minute given its mass m in kilograms.

a. A common shrew has a mass of only about 0.01 kg. To the nearest 10, what is the model's estimate for this shrew's resting heart rate?

b. What is the model's estimate for the resting heart rate of an American elk with a mass of 300 kg?

c. Two animal species differ in mass by a multiple of 10. According to the model, about what percent of the smaller animal's resting heart rate would you expect the larger animal's resting heart rate to be?

a. Substitute _____ for m in the formula and use a calculator.

$$h(m) = 241\left(\boxed{}\right)^{-\frac{1}{4}} \approx \boxed{}$$

The model estimates the shrew's resting heart rate to be about _____ beats per minute.

b. Substitute _____ for m in the formula and use a calculator.

$$h(m) = 241\left(\boxed{}\right)^{-\frac{1}{4}} \approx \boxed{}$$

The model estimates the elk's resting heart rate to be about _____ beats per minute.

c. Find the ratio of $h(m)$ for the _____ animal to the _____ animal. Let 1 represent the mass of the smaller animal.

$$\frac{241 \cdot \boxed{}^{-\frac{1}{4}}}{241 \cdot 1^{-\frac{1}{4}}} = \boxed{}^{-\frac{1}{4}} = \frac{1}{10^{\boxed{}}} \approx \boxed{}$$

You would expect the larger animal's resting heart rate to be about _____ of the smaller animal's resting heart rate.

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6. What is the difference between a power function and an exponential function?

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7. In Part B, the exponent is negative. Are the results consistent with the meaning of a negative exponent that you learned for integers? Explain.

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Your Turn

8. Use Kleiber's law from Part A.

   a. Find the basal metabolic rate for a 170 kilogram lion to the nearest 50 Calories. Then find the formula's prediction for a 70 kilogram human.

   b. Use your metabolic rate result for the lion to find what the basal metabolic rate for a 70 kilogram human _would_ be _if_ metabolic rate and mass were directly proportional. Compare the result to the result from Part a.

💬 Elaborate

9. Explain how can you use a radical to write and evaluate the power $4^{2.5}$.

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10. When $y = kx$ for some constant k, y varies directly as x. When $y = kx^2$, y varies directly as the square of x; and when $y = k\sqrt{x}$, y varies directly as the square root of x. How could you express the relationship $y = kx^{\frac{3}{5}}$ for a constant k?

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11. **Essential Question Check-In** Which of the following are true? Explain.

- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the nth root of a. Then raise the result to the mth power.
- To evaluate an expression of the form $a^{\frac{m}{n}}$, first find the mth power of a. Then find the nth root of the result.

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Translate expressions with rational exponents into radical expressions. Simplify numerical expressions when possible. Assume all variables are positive.

1. $64^{\frac{5}{3}}$

2. $x^{\frac{p}{q}}$

3. $(-512)^{\frac{2}{3}}$

4. $3^{\frac{2}{7}}$

5. $-\left(\dfrac{729}{64}\right)^{\frac{5}{6}}$

6. $0.125^{\frac{4}{3}}$

7. $vw^{\frac{2}{3}}$

8. $(-32)^{0.6}$

Translate radical expressions into expressions with rational exponents. Simplify numerical expressions when possible. Assume all variables are positive.

9. $\sqrt[7]{y^5}$

10. $\sqrt[7]{(-6)^6}$

11. $\sqrt[3]{3^{15}}$

**12.** $\sqrt[4]{(\pi z)^3}$

13. $\sqrt[6]{(bcd)^4}$

14. $\sqrt{6^6}$

15. $\sqrt[5]{32^2}$

16. $\sqrt[3]{\left(\dfrac{4}{x}\right)^9}$

17. **Music** Frets are small metal bars positioned across the neck of a guitar so that the guitar can produce the notes of a specific scale. To find the distance a fret should be placed from the bridge, multiply the length of the string by $2^{-\frac{n}{12}}$, where n is the number of notes higher than the string's root note. Where should a fret be placed to produce a F note on a B string (6 notes higher) given that the length of the string is 64 cm?

E string

Frets

Bridge

64 cm
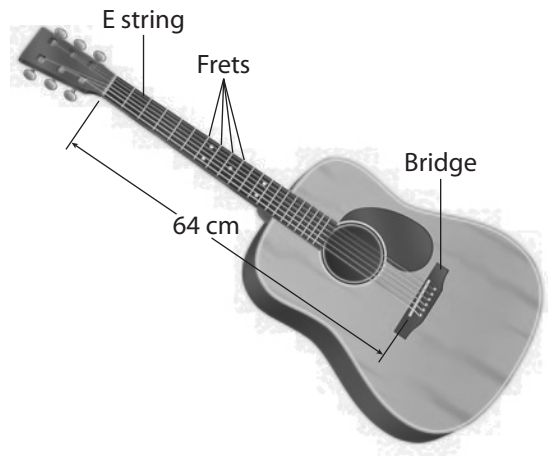
18. **Meteorology** The function $W = 35.74 + 0.6215T - 35.75V^{\frac{4}{25}} + 0.4275TV^{\frac{4}{25}}$ relates the windchill temperature W to the air temperature T in degrees Fahrenheit and the wind speed V in miles per hour. Use a calculator to find the wind chill temperature to the nearest degree when the air temperature is 28 °F and the wind speed is 35 miles per hour.

19. **Astronomy** New stars can form inside a cloud of interstellar gas when a cloud fragment, or *clump*, has a mass M greater than the *Jean's mass* M_J. The Jean's mass is $M_J = 100n^{-\frac{1}{2}}(T + 273)^{\frac{3}{2}}$ where n is the number of gas molecules per cubic centimeter and T is the gas temperature in degrees Celsius. A gas clump has $M = 137$, $n = 1000$, and $T = -263$. Will the clump form a star? Justify your answer.

20. Urban geography The total wages W in a metropolitan area compared to its total population p can be approximated by a power function of the form $W = a \cdot p^{\frac{9}{8}}$ where a is a constant. About how many times as great does the model predict the total earnings for a metropolitan area with 3,000,000 people will be as compared to a metropolitan area with a population of 750,000?

21. Which statement is true?

A. In the expression $8x^{\frac{3}{4}}$, $8x$ is the radicand.

B. In the expression $(-16)x^{\frac{4}{5}}$, 4 is the index.

C. The expression $1024^{\frac{n}{m}}$ represents the mth root of the nth power of 1024.

D. $50^{-\frac{2}{5}} = -50^{\frac{2}{5}}$

E. $\sqrt{(xy)^3} = xy^{\frac{3}{2}}$

> **H.O.T. Focus on Higher Order Thinking**

22. Explain the Error A teacher asked students to evaluate $10^{-\frac{3}{5}}$ using their graphing calculators. The calculator entries of several students are shown below. Which entry will give the incorrect result? Explain.

$10^{(-3/5)}$ $\sqrt[5]{10^{-3}}$ $10^{-.6}$ $1/10^{5/3}$ $(1/10^{1/5})^3$

23. Critical Thinking The graphs of three functions of the form $y = ax^{\frac{m}{n}}$ are shown for a specific value of a, where m and n are natural numbers. What can you conclude about the relationship of m and n for each graph? Explain.
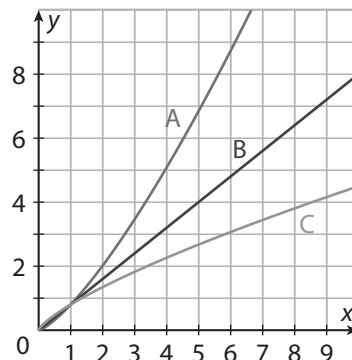
24. Critical Thinking For a negative real number a, under what condition(s) on m and n $\left(n \neq 0\right)$ is $a^{\frac{m}{n}}$ a real number? Explain. (Assume $\frac{m}{n}$ is written in simplest form.)

Lesson Performance Task

The formula $W = 35.74 + 0.6215T - 35.75V^{\frac{4}{25}} + 0.4275TV^{\frac{4}{25}}$ relates the wind chill temperature W to the air temperature T in degrees Fahrenheit and the wind speed V in miles per hour. Find the wind chill to the nearest degree when the air temperature is 40 °F and the wind speed is 35 miles per hour. If the wind chill is about 23 °F to the nearest degree when the air temperature is 40 °F, what is the wind speed to the nearest mile per hour?