Name ____________________ Class ____________ Date ________

10.1 Inverses of Simple Quadratic and Cubic Functions

Resource Locker

Essential Question: What functions are the inverses of quadratic functions and cubic functions, and how can you find them?

Explore Finding the Inverse of a Many-to-One Function

The function $f(x)$ is defined by the following ordered pairs: $(-2, 4)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, and $(2, 4)$.

(A) Find the inverse function of $f(x)$, $f^{-1}(x)$, by reversing the coordinates in the ordered pairs.

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(B) Is the inverse also a function? Explain.

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(C) If necessary, restrict the domain of $f(x)$ such that the inverse, $f^{-1}(x)$, is a function.

To restrict the domain of $f(x)$ so that its inverse is a function, you can restrict it to $\left\{x \mid x \geq \square \right\}$

(D) With the restricted domain of $f(x)$, what ordered pairs define the inverse function $f^{-1}(x)$?

____

Reflect

1. **Discussion** Look again at the ordered pairs that define $f(x)$. Without reversing the order of the coordinates, how could you have known that the inverse of $f(x)$ would not be a function?

____

2. How will restricting the domain of $f(x)$ affect the range of its inverse?

____

Explain 1 Finding and Graphing the Inverse of a Simple Quadratic Function

The function $f(x) = x^2$ is a many-to-one function, so its domain must be restricted in order to find its inverse function. If the domain is restricted to $x \geq 0$, then the inverse function is $f^{-1}(x) = \sqrt{x}$; if the domain is restricted to $x \leq 0$, then the inverse function is $f^{-1}(x) = -\sqrt{x}$.

The inverse of a quadratic function is a **square root function**, which is a function whose rule involves $\sqrt{x}$. **The parent square root function** is $g(x) = \sqrt{x}$. A square root function is defined only for values of x that make the expression under the radical sign nonnegative.

Example 1 **Restrict the domain of each quadratic function and find its inverse. Confirm the inverse relationship using composition. Graph the function and its inverse.**

(A) $f(x) = 0.5x^2$

Restrict the domain. $\left\{x \mid x \geq 0\right\}$

Find the inverse.

Replace $f(x)$ with y. $\quad y = 0.5x^2$

Multiply both sides by 2. $\quad 2y = x^2$

Use the definition of positive square root. $\quad \sqrt{2y} = x$

Switch x and y to write the inverse. $\quad \sqrt{2x} = y$

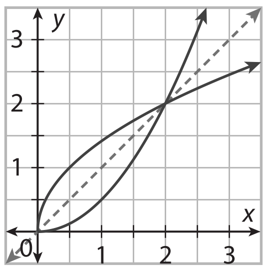
Replace y with $f^{-1}(x)$. $\quad f^{-1}(x) = \sqrt{2x}$

Confirm the inverse relationship using composition.

$$\begin{aligned} f^{-1}\big(f(x)\big) &= f^{-1}\big(0.5x^2\big) \\ &= \sqrt{2\big(0.5x^2\big)} \\ &= \sqrt{x^2} \\ &= x \text{ for } x \geq 0 \end{aligned} \qquad \begin{aligned} f\big(f^{-1}(x)\big) &= 0.5\left(\sqrt{2x}\right)^2 \\ &= 0.5(2x) \\ &= x \text{ for } x \geq 0 \end{aligned}$$

Since $f^{-1}\big(f(x)\big) = x$ for $x \geq 0$ and $f\big(f^{-1}(x)\big) = x$ for $x \geq 0$, it has been confirmed that $f^{-1}(x) = \sqrt{2x}$ for $x \geq 0$ is the inverse function of $f(x) = 0.5x^2$ for $x \geq 0$.

Graph $f^{-1}(x)$ by graphing $f(x)$ over the restricted domain and reflecting the graph over the line $y = x$.



(B) $f(x) = x^2 - 7$

Find the inverse.

Restrict the domain. ____________

Replace $f(x)$ with y. $\boxed{} = x^2 - 7$

Add 7 to both sides. $\boxed{} = x^2$

Use the definition of positive square root. $\boxed{} = x$

Switch x and y to write the inverse. ____________

Replace y with $f^{-1}(x)$. ____________

Confirm the inverse relationship using composition.

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\boxed{}\right) \qquad f\left(f^{-1}(x)\right) = f\left(\boxed{}\right)$$

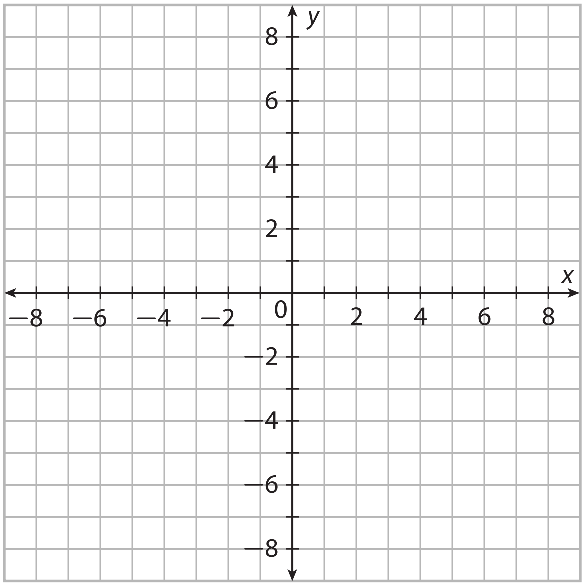
$$= \boxed{} \qquad = \left(\boxed{}\right)^2 - 7$$

$$= \boxed{} \qquad = \boxed{} - 7$$

$$= \boxed{} \qquad = \boxed{}$$

Since $f^{-1}\left(f(x)\right) = \boxed{}$ for ________ and $f\left(\boxed{}\right) = x$ for $x \geq \boxed{}$, it has been confirmed that $f^{-1}(x) = \boxed{}$ for $\boxed{}$ is the inverse function of $f(x) = x^2 - 7$ for ________.

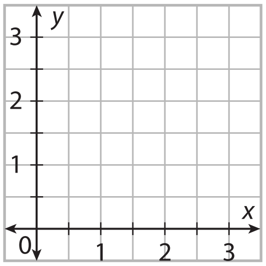
Graph $f^{-1}(x)$ by graphing $f(x)$ over the restricted domain and reflecting the graph over the line $y = x$.



Your Turn

Restrict the domain of each quadratic function and find its inverse. Confirm the inverse relationship using composition. Graph the function and its inverse.

3. $f(x) = 3x^2$



Explain 2 Finding the Inverse of a Quadratic Model

In many instances, quadratic functions are used to model real-world applications. It is often useful to find and interpret the inverse of a quadratic model. Note that when working with real-world applications, it is more useful to use the notation $x(y)$ for the inverse of $y(x)$ instead of the notation $y^{-1}(x)$.

Example 2 **Find the inverse of each of the quadratic functions. Use the inverse to solve the application.**

(A) The function $d(t) = 16t^2$ gives the distance d in feet that a dropped object falls in t seconds. Write the inverse function $t(d)$ to find the time t in seconds it takes for an object to fall a distance of d feet. Then estimate how long it will take a penny dropped into a well to fall 48 feet.

The original function $d(t) = 16t^2$ is a quadratic function with a domain restricted to $t \geq 0$.

Find the inverse function.

Write $d(t)$ as d. $\quad d = 16t^2$

Divide both sides by 16. $\quad \frac{d}{16} = t^2$

Use the definition of positive square root. $\quad \sqrt{\frac{d}{16}} = t$

Write t as $t(d)$. $\sqrt{\frac{d}{16}} = t(d)$

The inverse function is $t(d) = \sqrt{\frac{d}{16}}$ for $d \geq 0$.

Use the inverse function to estimate how long it will take a penny dropped into a well to fall 48 feet. Substitute $d = 48$ into the inverse function.

Write the function. $t(d) = \sqrt{\frac{d}{16}}$

Substitute 48 for d. $t(48) = \sqrt{\frac{48}{16}}$

Simplify. $t(48) = \sqrt{3}$

Use a calculator to estimate. $t(48) \approx 1.7$

So, it will take about 1.7 seconds for a penny to fall 48 feet into the well.

(B) The function $E(v) = 4v^2$ gives the kinetic energy E in Joules of an 8-kg object that is traveling at a velocity of v meters per second. Write and graph the inverse function $v(E)$ to find the velocity v in meters per second required for an 8-kg object to have a kinetic energy of E Joules. Then estimate the velocity required for an 8-kg object to have a kinetic energy of 60 Joules.

The original function $E(v) = 4v^2$ is a ______________ function with a domain restricted to v ______________.

Find the inverse function.

Write $E(v)$ as E. $\boxed{} = 4v^2$

Divide both sides by 4. $\boxed{} = v^2$

Use the definition of positive square root. $\boxed{} = v$

Write v as $v(E)$. ______________

The inverse function is $v(E) =$ ______________ for E ______________.

Use the inverse function to estimate the velocity required for an 8-kg object to have a kinetic energy of 60 Joules.

Substitute $E = 60$ into the inverse function.

Write the function. $v(E) = \boxed{}$

Substitute 60 for E. $v\left(\boxed{}\right) = \boxed{}$

Simplify. ______________

Use a calculator to estimate. ______________

So, an 8-kg object with kinetic energy of 60 Joules is traveling at a velocity of ______________ meters per second.

Your Turn

Find the inverse of the quadratic function. Use the inverse to solve the application.

4. The function $A(r) = \pi r^2$ gives the area of a circular object with respect to its radius r. Write the inverse function $r(A)$ to find the radius r required for area of A. Then estimate the radius of a circular object that has an area of 40 cm^2.

Explain 3 Finding and Graphing the Inverse of a Simple Cubic Function

Note that the function $f(x) = x^3$ is a one-to-one function, so its domain does not need to be restricted in order to find its inverse function. The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

The inverse of a cubic function is a **cube root function**, which is a function whose rule involves $\sqrt[3]{x}$. The **parent cube root function** is $g(x) = \sqrt[3]{x}$.

Example 3 **Find the inverse of each cubic function. Confirm the inverse relationship using composition. Graph the function and its inverse.**

(A) $f(x) = 0.5x^3$

Find each inverse. Graph the function and its inverse.

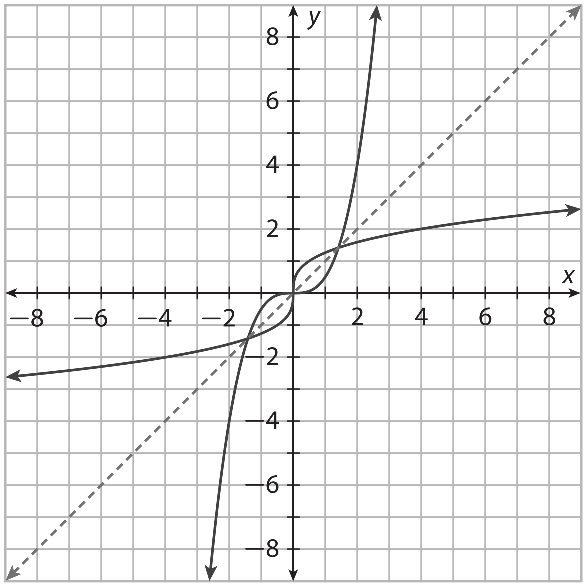
Replace $f(x)$ with y.	$y = 0.5x^3$
Multiply both sides by 2.	$2y = x^3$
Use the definition of cube root.	$\sqrt[3]{2y} = x$
Switch x and y to write the inverse.	$\sqrt[3]{2x} = y$
Replace y with $f^{-1}(x)$.	$\sqrt[3]{2x} = f^{-1}(x)$

Confirm the inverse relationship using composition.

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(0.5x^3) \\ &= \sqrt[3]{2(0.5x^3)} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned} \qquad \begin{aligned} f(f^{-1}(x)) &= f(\sqrt[3]{2x}) \\ &= 0.5(\sqrt[3]{2x})^3 \\ &= 0.5(2x) \\ &= x \end{aligned}$$

Since $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$, it has been confirmed that $f^{-1}(x) = \sqrt[3]{2x}$ is the inverse function of $f(x) = 0.5x^3$.

Graph $f^{-1}(x)$ by graphing $f(x)$ and reflecting $f(x)$ over the line $y = x$.



(B) $f(x) = x^3 - 9$

Find the inverse.

Replace $f(x)$ with y. $\boxed{} = x^3 - 9$

Add 9 to both sides. $\boxed{} = x^3$

Use the definition of cube root. $\boxed{} = x$

Switch x and y to write the inverse. ________________

Replace y with $f^{-1}(x)$. ________________

Confirm the inverse relationship using composition.

$$f^{-1}(f(x)) = f^{-1}\left(\boxed{}\right) \qquad f(f^{-1}(x)) = f\left(\boxed{}\right)$$

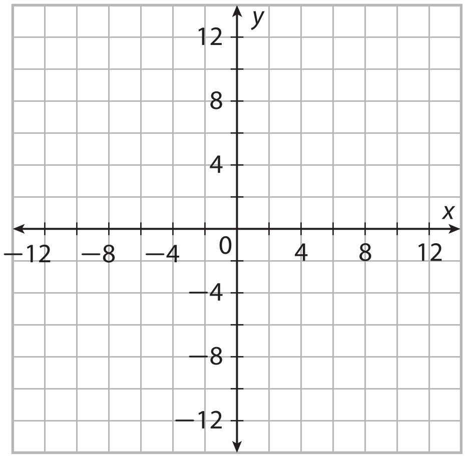
$$= \boxed{} \qquad = \boxed{}$$

$$= \boxed{} \qquad = \boxed{}$$

$$= \boxed{} \qquad = \boxed{}$$

Since $f^{-1}(f(x)) = \boxed{}$ and $f(f^{-1}(x)) = \boxed{}$, it has been confirmed that $f^{-1}(x) = \boxed{}$ is the inverse function of $f(x) = x^3 - 9$.

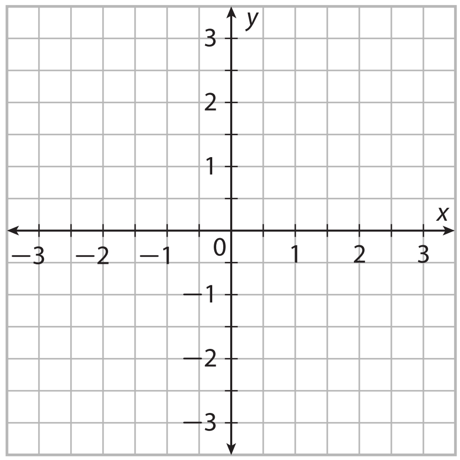
Graph $f^{-1}(x)$ by graphing $f(x)$ and reflecting $f(x)$ over the line $y = x$.



Your Turn

Find each inverse. Graph the function and its inverse.

5. $f(x) = 2x^3$



Explain 4 Finding the Inverse of a Cubic Model

In many instances, cubic functions are used to model real-world applications. It is often useful to find and interpret the inverse of cubic models. As with quadratic real-world applications, it is more useful to use the notation $x(y)$ for the inverse of $y(x)$ instead of the notation $y^{-1}(x)$.

Example 4 **Find the inverse of each of the following cubic functions.**

(A) The function $m(L) = 0.00001L^3$ gives the mass m in kilograms of a red snapper of length L centimeters. Find the inverse function $L(m)$ to find the length L in centimeters of a red snapper that has a mass of m kilograms.

The original function $m(L) = 0.00001L^3$ is a cubic function.

Find the inverse function.

Write $m(L)$ as m. $\quad m = 0.00001L^3$

Multiply both sides by 100,000. $\quad 100{,}000m = L^3$

Use the definition of cube root. $\quad \sqrt[3]{100{,}000m} = L$

Write L as $L(m)$. $\quad \sqrt[3]{100{,}000m} = L(m)$

The inverse function is $L(m) = \sqrt[3]{100,\,000m}$.

(B) The function $A(r) = \frac{4}{3}\pi r^3$ gives the surface area A of a sphere with radius r. Find the inverse function $r(A)$ to find the radius r of a sphere with surface area A.

The original function $A(r) = \frac{4}{3}\pi r^3$ is a ________ function.

Find the inverse function.

Write $A(r)$ as A. $\quad \boxed{} = \frac{4}{3}\pi r^3$

Divide both sides by $\frac{4}{3}\pi$. $\quad \boxed{} = r^3$

Use the definition of cube root. $\quad \boxed{} = r$

Write r as $r(A)$. ________

The inverse function is $r(A) =$ ________.

Your Turn

6. The function $m(r) = \frac{44}{3}\pi r^3$ gives the mass in grams of a spherical lead ball with a radius of r centimeters. Find the inverse function $r(m)$ to find the radius r of a lead sphere with mass m.

Elaborate

7. What is the general form of the inverse function for the function $f(x) = ax^2$? State any restrictions on the domains.

8. What is the general form of the inverse function for the function $f(x) = ax^3$? State any restrictions on the domains.

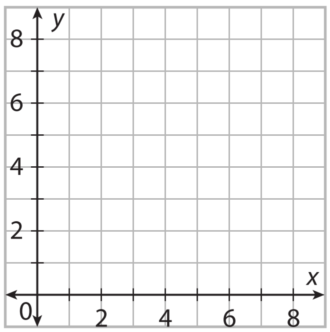
9. **Essential Question Check-In** Why must the domain be restricted when finding the inverse of a quadratic function, but not when finding the inverse of a cubic function?

Evaluate: Homework and Practice

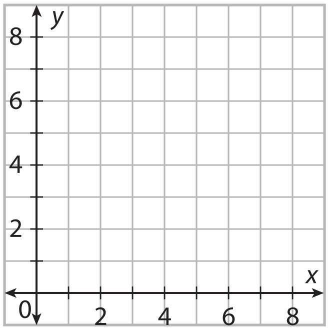
Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

Restrict the domain of the quadratic function and find its inverse. Confirm the inverse relationship using composition. Graph the function and its inverse.

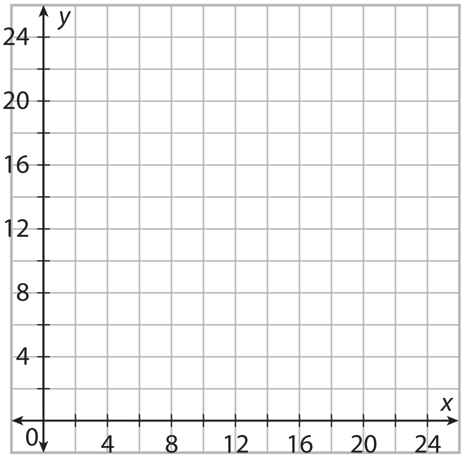
1. $f(x) = 0.2x^2$



2. $f(x) = 8x^2$



3. $f(x) = x^2 + 10$



Restrict the domain of the quadratic function and find its inverse. Confirm the inverse relationship using composition.

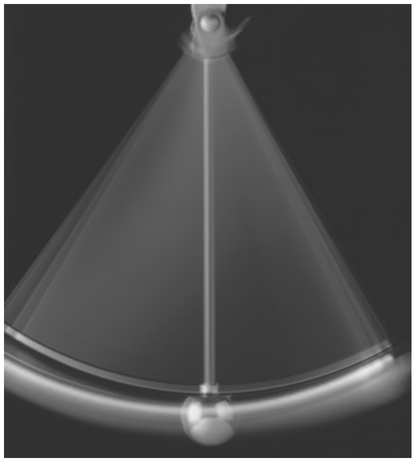
4. $f(x) = 15x^2$

5. $f(x) = x^2 - \frac{3}{4}$

6. $f(x) = 0.7x^2$

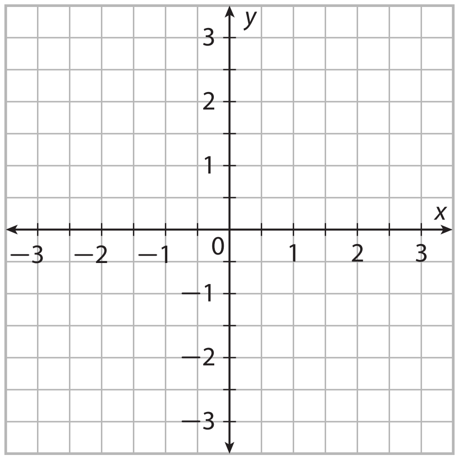
7. The function $d(s) = \frac{1}{14.9}s^2$ models the average depth d in feet of the water over which a tsunami travels, where s is the speed in miles per hour. Write the inverse function $s(d)$ to find the speed required for a depth of d feet. Then estimate the speed of a tsunami over water with an average depth of 1500 feet.

8. The function $x(T) = 9.8\left(\frac{T}{2\pi}\right)^2$ gives the length x in meters for a pendulum to swing for a period of T seconds. Write the inverse function to find the period of a pendulum in seconds. The period of a pendulum is the time it takes the pendulum to complete one back-and-forth swing. Find the period of a pendulum with length of 5 meters.

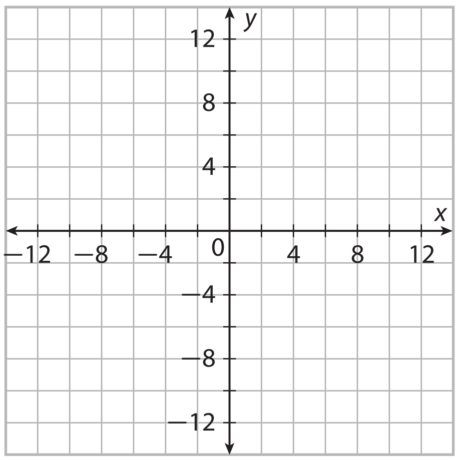


Find the inverse of each cubic function. Confirm the inverse relationship using composition. Graph the function and its inverse.

9. $f(x) = 0.25x^3$



10. $f(x) = -12x^3$



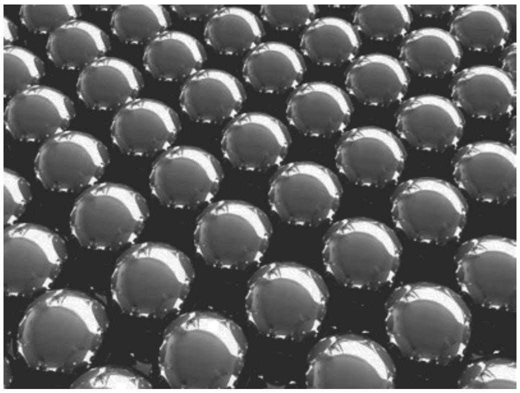
Find the inverse of the cubic function. Confirm the inverse relationship using composition.

11. $f(x) = x^3 - \frac{5}{6}$

12. $f(x) = x^3 + 9$

13. The function $m(r) = 31r^3$ models the mass in grams of a spherical zinc ball as a function of the ball's radius in centimeters. Write the inverse model to represent the radius r in cm of a spherical zinc ball as a function of the ball's mass m in g.

14. The function $m(r) = 21r^3$ models the mass in grams of a spherical titanium ball as a function of the ball's radius in centimeters. Write the inverse model to represent the radius r in centimeters of a spherical titanium ball as a function of the ball's mass m in grams.



15. The weight w in pounds that a shelf can support can be modeled by $w(d) = 82.9d^3$ where d is the distance, in inches, between the supports for the shelf. Write the inverse model to represent the distance d in inches between the supports of a shelf as a function of the weight w in pounds that the shelf can support.

H.O.T. Focus on Higher Order Thinking

16. Explain the Error A student was asked to find the inverse of the function $f(x) = \left(\frac{x}{2}\right)^3 + 9$. What did the student do wrong? Find the correct inverse.

$$f(x) = \left(\frac{x}{2}\right)^3 + 9$$
$$y = \left(\frac{x}{2}\right)^3 + 9$$
$$y - 9 = \left(\frac{x}{2}\right)^3$$
$$2y - 18 = x^3$$
$$\sqrt[3]{2y - 18} = x$$
$$y = \sqrt[3]{2x - 18}$$
$$f^{-1}(x) = \sqrt[3]{2x - 18}$$

17. Multi-Step A framing store uses the function $\left(\frac{c - 0.2}{0.5}\right)^2 = a$ to determine the total area of a piece of glass with respect to the cost before installation of the glass. Write the inverse function for the cost c in dollars of glass for a picture with an area of a in square centimeters. Then write a new function to represent the total cost C the store charges if it costs \$6.00 for installation. Use the total cost function to estimate the cost if the area of the glass is 192 cm^2.

18. **Make a Conjecture** The function $f(x) = x^2$ must have its domain restricted to have its inverse be a function. The function $f(x) = x^3$ does not need to have its domain restricted to have its inverse be a function. Make a conjecture about which power functions need to have their domains restricted to have their inverses be functions and which do not.

Lesson Performance Task

One method used to irrigate crops is the center-pivot irrigation system. In this method, sprinklers rotate in a circle to water crops. The challenge for the farmer is to determine where to place the pivot in order to water the desired number of acres. The farmer knows the area but needs to find the radius of the circle necessary to define that area. How can the farmer determine this from the formula for the area of a circle $A = \pi r^2$? Find the formula the farmer could use to determine the radius necessary to irrigate a given number of acres, A. (Hint: One acre is 43,560 square feet.) What would be the radius necessary for the sprinklers to irrigate an area of 133 acres?